ABOUT LIE GROUPS

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CONTENTS

NOTATIONAL NOTES

If M and N are smooth manifolds and $f\colon M \to N$ is a smooth map between them, we denote the induced map on tangent bundles by $Tf\colon TM \to TN$. For each $p \in M$, the linear map between tangent spaces induced by f is denoted $T_pf\colon T_pM \to T_{f(p)}N$. In this notation, Tf is defined by

$$(Tf)(p, \vec{v}) = (f(p), (T_pf)(\vec{v}))$$

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for each $(p, \vec{v}) \in TM$. This notation is to distinguish between the ***derivative*** Tf of a smooth map f, which is a map of tangent bundles, from the ***differential*** df of a smooth real-valued function f, which is a smooth 1-form.

(If $f\colon M \to \mathbb{R}$ is smooth, recall the relationship between Tf and df. If $p \in M$ and $\vec{v} \in T_pM$, then $(df)_p(\vec{v})$ is defined to be the unique real number such that

$$(T_pf)(\vec{v}) = (df)_p(\vec{v}) \cdot \frac{\partial}{\partial t}\Big|_{t=f(p)},$$

where $\frac{\partial}{\partial t}$ denotes the canonical positively-oriented unit length vector field on $\mathbb{R}$.)

We consider a vector field v on a smooth manifold M to be a function $v\colon M \to TM$ from the manifold to its tangent bundle. A point in the tangent bundle is a pair consisting of a point in the manifold and a tangent vector at that point. A vector field is required to be a ***section*** of the tangent bundle, meaning that when evaluated at a point $p \in M$, that point is the first entry in the pair specified by the element $v(p) \in TM$. We always use the notation

$$v(p) = (p, v_p) \in TM,$$

, where v is a vector field on M, $p \in M$, and $v_p \in T_pM$. These same constructions lead to similar notation for differential forms and arbitrary tensor fields on the manifold, which are sections of various other vector bundles over M. For example, the differential df of a smooth real-valued function $f\colon M \to \mathbb{R}$ is a section of the ***cotangent bundle*** T^*M of M, and so for each $p \in M$ we use the notation

$$df(p) = (p, (df)_p).$$

1. Lie Groups

An excellent source for most of the material presented here, and much more besides, is [2]. Their notation even agrees with ours!

1.1. **Beginning Details.**

Definition 1.1. A ***Lie group*** is a group G with the structure of a smooth manifold, such that the inversion and multiplication maps

$$G \to G, x \mapsto x^{-1} \qquad \text{and} \qquad G \times G \to G, (x, y) \mapsto xy$$

are smooth. It can be shown that it suffices to assume that multiplication is smooth. △

A Lie group G comes with a lot of structure. There is a distinguished element $e \in G$, the group identity. This means there is a distinguished tangent space, $T_eG \subset TG$. For each $g \in G$, we obtain three maps:

(1) ***left multiplication***: $L_g \colon G \to G$, $h \mapsto gh$;
(2) ***right multiplication***: $R_g \colon G \to G$, $h \mapsto hg$; and
(3) ***conjugation***: $\Psi_g \colon G \to G$, $h \mapsto L_g \circ R_{g^{-1}}(h) = R_{g^{-1}} \circ L_g(h) = ghg^{-1}$.

Notice that L_g and Ψ_g are group homomorphisms, but R_g is a group ***anti-homomorphism***. All three of these maps are smooth, and in fact they are all diffeomorphisms. The inverses of L_g, R_g, and Ψ_g are $L_{g^{-1}}$, $R_{g^{-1}}$, and $\Psi_{g^{-1}}$, respectively. Since L_g and Ψ_g are also group homomorphisms, they are ***Lie group isomorphisms***. Also, note that the left and right multiplication maps commute with each other: for all $g, h \in G$, we have $L_g \circ R_h = R_h \circ L_g$.

Remark 1.2. For whatever reason, most of Lie theory is centered around the left multiplication maps, but it could just as well have been developed using the right multiplication maps. ♢

The three maps above are ***canonical*** with respect to the Lie group structure. Therefore all tangent spaces of G are canonically isomorphic. For $g, h \in G$, we have the canonical linear isomorphisms

$$T_g(L_{hg^{-1}}) \colon T_gG \to T_hG \qquad \text{and} \qquad T_g(L_{gh^{-1}}) \colon T_hG \to T_gG.$$

Thus all tangent spaces of G are canonically isomorphic to the distinguished tangent space of G, T_eG.

Let $g \in G$. Because $\Psi_g(e) = geg^{-1} = e$, we have a canonical operator on the distinguished tangent space T_eG, given by $T_e\Psi_g \colon T_eG \to T_eG$. We denote this map by Ad_g. Since Ψ_g is a diffeomorphism, $\mathrm{Ad}_g = T_e\Psi_g$ is a linear isomorphism, so $\mathrm{Ad}_g \in \mathrm{GL}(T_eG)$, so

$$\mathrm{Ad} \colon G \to \mathrm{GL}(T_eG)$$

is a group representation of G, called the ***adjoint representation***.

Recall that $\mathrm{GL}(T_eG)$ is the inverse image of the open set $\mathbb{R} \setminus \{0\}$ under the continuous (and smooth) map $\det \colon \mathfrak{gl}(T_eG) \to \mathbb{R}$, so it is an open subset of the vector space $\mathfrak{gl}(T_eG)$. Thus $\mathrm{GL}(T_eG)$ is a smooth manifold with tangent bundle $\mathrm{GL}(T_eG) \times \mathfrak{gl}(T_eG)$. Therefore the tangent map of Ad at the identity e is a map

$T_e \mathrm{Ad}\colon T_eG \to \mathfrak{gl}(T_eG)$. By slightly restructuring the domain and codomain, we obtain a map

$$\mathrm{ad}\colon T_eG \times T_eG \to T_eG, \quad (X, Y) \mapsto \mathrm{ad}_X(Y) := (T_e \mathrm{Ad})(X)\, Y.$$

Note that ad is linear in both X and Y, so ad is bilinear.

1.2. **The Exponential Map and Useful Curves.**

For each Lie group G, we have the ***exponential map***, $\exp_G\colon T_eG \to G$. We usually omit the subscript from exp if there is no confusion. It is defined by means of ***one-parameter subgroups***, which we will not discuss here. The exponential map is characterized by the fact that if $X \in T_eG$ and $s, t \in \mathbb{R}$, then

$$\exp\left((s+t)X\right) = \exp(sX) \cdot \exp(tX) \left(= \exp(tX) \cdot \exp(sX)\right),$$

and the following Lemma.

Lemma 1.3. *Let $X \in T_eG$, and let $c\colon \mathbb{R} \to G$ be the smooth curve given by $t \mapsto \exp(tX)$. Then $c'(0) = X$.*

(Recall that the ***derivative*** c' of a smooth curve c is defined by

$$c'(t) := T_{c(t)}c \left(\left.\frac{\partial}{\partial s}\right|_{s=t} \right),$$

where $\frac{\partial}{\partial s}$ denotes the canonical positively-oriented unit length vector field on $\mathbb{R}$.)

Note that $\mathrm{GL}(T_eG)$ is a Lie group under multiplication, and that its tangent space at the identity is essentially $\mathfrak{gl}(T_eG)$. Therefore we have a map

$$\exp_{\mathrm{GL}(T_eG)}\colon \mathfrak{gl}(T_eG) \to \mathrm{GL}(T_eG).$$

Since $\mathrm{ad}_X \in \mathfrak{gl}(T_eG)$ for all $X \in T_eG$, we have

$$\exp_{\mathrm{GL}(T_eG)}(\mathrm{ad}_X) \in \mathrm{GL}(T_eG).$$

It is natural to ask what element of $\mathrm{GL}(T_eG)$ this might be.

Theorem 1.4. *Let $X \in T_eG$. Then*

$$\mathrm{Ad}_{\exp_G(X)} = \exp_{\mathrm{GL}(T_eG)}(\mathrm{ad}_X).$$

Remark 1.5. Dropping the subscripts from the exponential maps, we obtain the commutative diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{ad}} & \mathfrak{gl}(T_eG) \\ {\scriptstyle \exp}\big\downarrow & & \big\downarrow{\scriptstyle \exp} \\ G & \xrightarrow[\mathrm{Ad}]{} & GL(T_eG) \end{array}$$

◊

Combining Theorem 1.4 and Lemma 1.3 yields the following result.

Proposition 1.6. *Let* $X, Y \in T_eG$*, and let* $c\colon \mathbb{R} \to T_eG$ *be the smooth curve given by* $t \mapsto \mathrm{Ad}_{\exp(tX)}(Y)$*. Then*

$$c'(0) = (\mathrm{ad}_X(Y)) \,.$$

2. The Lie Algebra of a Lie Group

2.1. **General Lie Algebras.**

Definition 2.1. A ***Lie algebra*** is a real vector space L equipped with a skew-symmetric bilinear map $L \times L \to L, \quad (v, w) \mapsto [v, w]$, called a ***bracket***, which satisfies the ***Jacobi identity***:

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$$

for all $u, v, w \in L$. △

Two standard examples are the set of vector fields on a manifold with the Lie bracket, or the set of $n \times n$ real (or complex) matrices with the bracket

$$[A, B] := AB - BA.$$

2.2. **The Tangent Space at the Identity.**

The tangent space T_eG of G at the identity is a real vector space. Using the three classes of maps inherent in the Lie group structure, we can equip T_eG with a bracket that makes it a Lie algebra. The vector space T_eG with this bracket is denoted $\mathfrak{g}$, and called the ***Lie algebra of the Lie group*** G.

Each step in the construction of the Lie bracket for $\mathfrak{g}$ is *natural*, in the sense that it is preserved by smooth homomorphisms between Lie groups. Let H be another

Lie group and $\rho\colon G \to H$ be a smooth homomorphism. The naturality of each step below will be shown by a commutative diagram involving G, H, and ρ.

$$\begin{array}{ccc} G & \xrightarrow{\rho} & H \\ {\scriptstyle \Psi_g}\downarrow & & \downarrow{\scriptstyle \Psi_{\rho(g)}} \\ G & \xrightarrow[\rho]{} & H \end{array}$$

As described above, for each $g \in G$, we obtain a Lie group isomorphism $\Psi_g\colon G \to G$ and a linear isomorphism $\mathrm{Ad}_g\colon T_eG \to T_eG$. Then $\mathrm{Ad}_g \in GL(T_eG)$, so we have a smooth homomorphism $\mathrm{Ad}\colon G \to GL(T_eG)$.

$$\begin{array}{ccc} T_eG & \xrightarrow{T_e\rho} & T_eH \\ {\scriptstyle \mathrm{Ad}_g}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}(\rho(g))} \\ T_eG & \xrightarrow[T_e\rho]{} & T_eH \end{array}$$

We define a bracket on T_eG by $[X, Y] = \mathrm{ad}_X(Y)$. It remains to be shown that this bracket is anti-symmetric and satisfies the Jacobi identity. We will not prove this here, although it will follow from the fact that the Lie bracket of vector fields satisfies these properties.

$$\begin{array}{ccc} T_eG & \xrightarrow{T_e\rho} & T_eH \\ {\scriptstyle \mathrm{ad}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{ad}_{(T_e\rho)(X)}} \\ T_eG & \xrightarrow[T_e\rho]{} & T_eH \end{array}$$

2.3. **Left-Invariant Vector Fields.**

Definition 2.2. Let $f\colon M \to N$ be a diffeomorphism between smooth manifolds, and let $v \in \mathrm{Vec}(M)$ and $w \in \mathrm{Vec}(N)$. The ***pushforward*** of v by f is defined by

$$(f_*v)_q := \left(T_{f^{-1}(q)}f\right)\left(v_{f^{-1}(q)}\right)$$

for $q \in N$, and the ***pullback*** of w by f is defined by

$$(f^*w)_p := \left(T_{f(p)}f^{-1}\right)\left(w_{f(p)}\right)$$

for $p \in M$. Note that

$$f^* w = (f^{-1})_* w \qquad \text{and} \qquad f_* v = (f^{-1})^* v.$$

$\triangle$

Definition 2.3. A vector field $v \in \mathrm{Vec}(G)$ is called ***left-invariant*** if

$$(T_h L_g)(v_h) = v_{L_g(h)} = v_{gh}$$

for all $g, h \in G$. This means the following diagram commutes for each $g \in G$.

$$\begin{array}{ccc} G & \xrightarrow{v} & TG \\ {\scriptstyle L_g}\downarrow & & \downarrow{\scriptstyle TL_g} \\ G & \xrightarrow[v]{} & TG \end{array}$$

We denote the set of all left-invariant vector fields on G by $L(G)$. $\triangle$

Remark 2.4. Let $v \in L(G)$. Then $TL_g \circ X = X \circ L_g$ for all $g \in G$. Thus

$$TL_g \circ X \circ (L_g)^{-1} = X \qquad \text{and} \qquad (TL_g)^{-1} \circ X \circ L_g = X$$

for all $g \in G$. Certainly if a vector field satisfies either of the above equations for all $g \in G$ it must be left-invariant. Therefore $L(G)$ is the set of vector fields invariant under pushforward by left multiplication, which is also the set of vector fields invariant under pullback by left multiplication. $\lozenge$

The set $L(G)$ is clearly a real vector space, but it is not clear what its dimension is. There's no reason to assume that the dimension be finite, but it is. It's actually quite a surprise.

Recall the Lie bracket of vector fields. This can be defined in terms of flows of vector fields, or in terms of derivations. Let $v, w \in \mathrm{Vec}(G)$ be vector fields, let Φ_v^t, Φ_w^t denote their respective flows, and let $\mathcal{D}_v, \mathcal{D}_w$ denote their respective associated derivations. Then the ***Lie bracket*** $[v, w] \in \mathrm{Vec}(G)$ is the unique vector field such that

$$[v, w] = \mathcal{L}_v w := \frac{d}{dt} (\Phi_v^t)^* w \Big|_{t=0},$$

or equivalently,

$$\mathcal{D}_{[v,w]} = \mathcal{D}_v \circ \mathcal{D}_w - \mathcal{D}_w \circ \mathcal{D}_v.$$

The pushforward of vector fields by diffeomorphisms preserves the Lie bracket. (See page 144 in [1].) Since left-invariant vector fields can be categorized as those that are invariant under pushforward by all left multiplications, this implies that the Lie bracket of two left-invariant vector fields is also left-invariant. Therefore $L(G)$ equipped with the Lie bracket of vector fields is a Lie algebra.

2.4. $T_eG \cong L(G)$ as Vector Spaces.

We have two Lie algebras associated with G: the tangent space at the identity, T_eG, with the bracket induced by ad, and the left-invariant vector fields, $L(G)$, with the Lie bracket. In this section we will demonstrate that they are isomorphic as vector spaces.

Define a map $\nu \colon T_eG \to \mathrm{Vec}(G)$ by

$$\nu(X)_g = T_eL_g(X)$$

for all $X \in T_eG$ and $g \in G$. Because tangent maps are linear, so is ν. For all $X \in T_eG$ and $g, h \in G$ we have

$$(T_hL_g)\,(\mathbf{v}(X)_h) = (T_hL_g)\,(T_eL_h(X)) = T_e(L_g \circ L_h)(X) = T_eL_{gh}(X) = \nu(X)_{gh} = \nu(X)_{L_g(h)}.$$

Therefore $\nu(X)$ is left invariant, so ν really is a map $T_eG \to L(G)$. Its inverse is (immediately) given by the map

$$L(G) \to T_eG, \qquad v \mapsto v_e \in T_eG.$$

2.5. $T_eG \cong L(G)$ as Lie Algebras.

To show that T_eG and $L(G)$ are isomorphic as Lie algebras as well as vector fields, we must show that the map

$$\nu \colon T_eG \to L(G)$$

preserves the brackets, i.e.

$$\nu(\mathrm{ad}_X Y) = [\nu(X), \nu(Y)]$$

for all $X, Y \in T_eG$. Since the Lie bracket of vector fields can be described easily in terms of flows, it might be helpful to know what the flows of these vector fields look like.

Claim 2.5. *Let $X \in T_eG$ and $g \in G$. Then the flow of $\nu(X)$ through g is the curve $c \colon \mathbb{R} \to G$ given by*

$$c(t) = g \cdot \exp(tX).$$

Proof. Note first that $c(0) = g \cdot \exp(\vec{0}) = ge = g$. Now, let $t \in \mathbb{R}$. Then

$$\begin{aligned}
c'(t) &= \frac{d}{ds}\Big|_{s=t} c(t) = \frac{d}{ds}\Big|_{s=0} c(s+t) \\
&= \frac{d}{ds}\Big|_{s=0} g \cdot \exp\left((s+t)X\right) \\
&= \frac{d}{ds}\Big|_{s=0} g \cdot \exp tX \cdot \exp sX \\
&= \frac{d}{ds}\Big|_{s=0} L_{g\cdot\exp tX}\left(\exp sX\right) \\
&= \left(T_e L_{g\cdot\exp tX}\right)\left(\frac{d}{ds}\Big|_{s=0} \exp sX\right) \\
&= \left(T_e L_{g\cdot\exp tX}\right)(X) \\
&= \nu(X)_{g\cdot\exp tX} \\
&= \nu(X)_{c(t)}.
\end{aligned}$$

QED

Theorem 2.6. *Let* $X, Y \in T_e G$. *Then*

$$\nu\left(\mathrm{ad}_X(Y)\right) = \left[\nu(X), \nu(Y)\right].$$

Proof. From Claim 2.5, we know that the flow of $\nu(X)$ at time $t \in \mathbb{R}$ is the map $G \to G$ given by $R_{\exp(tX)}$. Let $g \in G$. From the definition of the Lie bracket of vector fields in terms of Lie derivatives, we have

$$\left[\nu(X), \nu(Y)\right]_g = \left(\mathcal{L}_{\nu(X)}\left(\nu(Y)\right)\right)_g = \frac{d}{dt}\Big|_{t=0}\left(\left(R_{\exp tX}\right)^* \nu(Y)\right)_g.$$

By the definitions of the pullback of vector fields and $\nu(Y)$, the properties of left and right multiplication, and the chain rule, we have

$$\begin{aligned}
\left((R_{\exp tX})^*\nu(Y)\right)_g &= \left(T_{g\cdot\exp tX}(R_{\exp tX})^{-1}\right)\left(\nu(Y)_{g\cdot\exp tX}\right) \\
&= T_{g\cdot\exp tX}(R_{\exp tX})^{-1} \circ T_e(L_{g\cdot\exp tX})(Y) \\
&= T_e\left((R_{\exp tX})^{-1} \circ L_{g\cdot\exp tX}\right)(Y) \\
&= T_e\left(R_{(\exp tX)^{-1}} \circ L_g \circ L_{\exp tX}\right)(Y) \\
&= T_e\left(L_g \circ L_{\exp tX} \circ R_{(\exp tX)^{-1}}\right)(Y) \\
&= T_e\left(L_g \circ \Psi_{\exp tX}\right)(Y) \\
&= (T_eL_g) \circ \left(T_e\Psi_{\exp tX}\right)(Y) \\
&= (T_eL_g)\left(\mathrm{Ad}_{\exp tX}(Y)\right).
\end{aligned}$$

Since T_eL_g is a linear map, it commutes with $\frac{d}{dt}$. Using this and Proposition 1.6, we conclude that

$$\begin{aligned}
[\nu(X),\nu(Y)]_g &= \left.\frac{d}{dt}\right|_{t=0}\left((R_{\exp tX})^*\nu(Y)\right)_g \\
&= \left.\frac{d}{dt}\right|_{t=0}(T_eL_g)\left(\mathrm{Ad}_{\exp tX}(Y)\right) \\
&= (T_eL_g)\left(\left.\frac{d}{dt}\right|_{t=0}\mathrm{Ad}_{\exp tX}(Y)\right) \\
&= (T_eL_g)\left(\mathrm{ad}_X(Y)\right) \\
&= \nu\left(\mathrm{ad}_X(Y)\right)_g,
\end{aligned}$$

as desired. QED

References

[1] Lawrence Conlon. *Differentiable manifolds*. Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Boston Inc., Boston, MA, second edition, 2001.

[2] J.J. Duistermaat and J.A.C. Kolk. *Lie Groups (Universitext)*. Springer, 2000.