THE VERY, VERY BASICS OF HAMILTONIAN ACTIONS ON SYMPLECTIC MANIFOLDS

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Notational Notes

If M and N are smooth manifolds and $f\colon M \to N$ is a smooth map between them, we denote the induced map on tangent bundles by $Tf\colon TM \to TN$. For each $p \in M$, the linear map between tangent spaces induced by f is denoted $T_pf\colon T_pM \to T_{f(p)}N$. This notation is to distinguish between the **derivative** Tf of a smooth map f, which is a map of tangent bundles, from the **differential** df of a smooth real-valued function f, which is a smooth 1-form.

(If $f\colon M \to \mathbb{R}$ is smooth, recall the relationship between Tf and df. If $p \in M$ and $\vec{v} \in T_pM$, then $(df)_p(\vec{v})$ is defined to be the unique real number such that

$$(T_pf)(\vec{v}) = (df)_p(\vec{v}) \cdot \frac{\partial}{\partial t}\Big|_{t=f(p)},$$

where $\frac{\partial}{\partial t}$ denotes the canonical positively-oriented unit length vector field on $\mathbb{R}$.)

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1. Smooth group actions

Let G be a Lie group and M be a smooth manifold.

Definition 1.1. A **smooth action** of G on M is a group homomorphism

$$G \to \mathrm{Diff}(M), \; g \mapsto \mathcal{A}_g$$

such that the associated map

$$\mathcal{A}\colon G \times M \to M, \; (g, x) \mapsto \mathcal{A}_g(x)$$

is smooth. We sometimes write $g \cdot x = A_g(x)$ if there is no confusion.

Notice that for each $g \in G$, the map $\mathcal{A}_g\colon M \to M$ really is a diffeomorphism, because $\mathcal{A}_{g^{-1}}$ is its inverse.

Recall that the **Lie algebra** $\mathfrak{g}$ of G can be defined as $\mathfrak{g} := T_1 G$, the tangent space to G at the identity $1 \in G$. Therefore each element of $\mathfrak{g}$ represents an "infinitesimal nudge" away from 1 in G. Since the element $1 \in G$ acts on M as the identity map, an infinitesimal nudge in G away from 1 gives rise to an infinitesimal nudge in M away from each element, *which is exactly what a vector field on* M *is*!

Definition 1.2. For each $\xi \in \mathfrak{g}$, the **fundamental vector field** on M induced by ξ is the vector field ξ_M defined by

$$(\xi_M)_x := \frac{d}{dt} \mathcal{A}_{\exp(t\xi)}(x) \bigg|_{t=0}$$

for each $x \in M$.

The idea behind this definition is that we start with the curve $t \mapsto \exp(t\xi)$ in G, which has velocity ξ at time $t = 0$, and transport it to M via the action $\mathcal{A}$, obtaining for each point $x \in M$ the curve $t \mapsto \mathcal{A}_{\exp(t\xi)}(x)$. The velocity of this last curve at time $t = 0$ is $(\xi_M)_x$.

Fact 1.3.

(1) For each $\xi \in \mathfrak{g}$, the induced vector field ξ_M on M is smooth.

(2) For each $x \in M$, define $\mathcal{A}^x \colon G \to M$ by $\mathcal{A}^x(g) = \mathcal{A}_g(x)$. Then

$$(\xi_M)_x := (T_1 \mathcal{A}^x)(\xi)$$

for each $\xi \in \mathfrak{g}$ and $x \in M$.

2. Symplectic manifolds

Let M be a smooth manifold.

Definition 2.1. A **symplectic structure** on M is a smooth differential 2-form ω that is closed and non-degenerate. A **symplectic manifold** is a pair (M, ω) consisting of a smooth manifold M and a choice of symplectic structure ω on it.

Recall that a smooth differential 2-form ω is a smoothly-varying collection $\omega = \{\omega_x \mid x \in M\}$ of **skew-symmetric bilinear forms**,

$$\omega_x \colon T_xM \times T_xM \to \mathbb{R}.$$

That ω is **closed** means that its exterior derivative is zero, $d\omega = 0$. This condition can be viewed as an *integrability condition* in an extremely cool way, which I will not go into here. (See [G].)

That ω is **non-degenerate** means that each bilinear form ω_x is non-degenerate, meaning that for each nonzero vector $\vec{u} \in T_xM$ there exists a vector $\vec{v} \in T_xM$ such that $\omega_x(\vec{u}, \vec{v}) \neq 0$.

A useful equivalent definition for the bilinear form ω_x to be non-degenerate involves the associated linear map

$$\widetilde{\omega}_x \colon T_xM \to (T_xM)^* = T_x^*M$$

defined by "plugging into the first slot": $\widetilde{\omega}_x(\vec{u}) = \omega_x(\vec{u}, \cdot)$ for each $\vec{u} \in T_xM$. The bilinear form ω_x is non-degenerate if and only if the linear map $\widetilde{\omega}_x$ is an isomorphism.

Taking things to the manifold level, the association of a linear map $\widetilde{\omega}_x$ to each bilinear form ω_x allows us to define a smooth bundle map between the tangent and cotangent bundles, $\widetilde{\omega} \colon TM \to T^*M$. The differential 2-form ω is non-degenerate if and only if $\widetilde{\omega}$ is a bundle isomorphism.

Let $f\colon M \to N$ be a smooth map between manifolds. Recall that f defines a **pull-back map** f^* from differential forms on N to differential forms on M. Specifically, if α is a differential k-form on N, then $f^*\alpha$ is the k-form on M defined by

$$(f^*\alpha)_x(\vec{v}_1, \ldots, \vec{v}_k) := \alpha_{f(x)}\left((T_x f)\,\vec{v}_1, \ldots, (T_x f)\,\vec{v}_k\right)$$

for each $x \in M$ and $\vec{v}_1, \ldots, \vec{v}_k \in T_x M$.

Definition 2.2. Let (M, ω) and (N, σ) be symplectic manifolds. A smooth map $f\colon M \to N$ is **symplectic** if $f^*\sigma = \omega$. The map f is a **symplectomorphism** if it is a symplectic diffeomorphism.

3. Symplectic and Hamiltonian vector fields

Let (M, ω) be a symplectic manifold.

Definition 3.1. Let $X \in \mathrm{Vec}(M)$, and let $\phi_t\colon M \to M$ denote the time t flow on M in the direction X. The vector field X is **symplectic** if ϕ_t is a symplectomorphism for each t for which it is defined.

Proposition 3.2. *Let $X \in \mathrm{Vec}(M)$, and let $\phi_t\colon M \to M$ be the time t flow of X on M. The following are equivalent.*

(1) $(\phi_t)^*\omega = \omega$ *for all* t, *(i.e.* X *is symplectic);*
(2) $\mathcal{L}_X\omega = 0$*;*
(3) $\iota_X\omega$ *is closed.*

(Here $\mathcal{L}_X$ denotes the Lie derivative in the direction X, and $\iota_X\omega = \omega(X, \cdot)$.)

Proof. It's not too hard to see that statements (1) and (2) are equivalent, since one is obtained immediately from the other by differentiation or integration. We will prove that (2) is equivalent to (3).

By Cartan's identity, we know that

$$\mathcal{L}_X = d \circ \iota_X + \iota_X \circ d.$$

Because the symplectic form ω is by definition closed, we have

$$\mathcal{L}_X\omega = d\left(\iota_X\omega\right) + \iota_X\,d\omega = d\left(\iota_X\omega\right) + \iota_X(0) = d\left(\iota_X\omega\right).$$

Thus statements (2) and (3) are equivalent. □

Recall the bundle isomorphism $\widetilde{\omega}\colon TM \to T^*M$ induced by the symplectic form ω on M. This $\widetilde{\omega}$ induces an isomorphism between smooth sections of these two bundles, which are vector fields and differential 1-forms, respectively, defined by

$$X \mapsto \widetilde{\omega} \circ X = \iota_X \omega$$

for each $X \in \mathrm{Vec}(M)$. At a particular point $x \in M$, the covector $\widetilde{\omega}(X_x) \in T_x^*M$ associated to the vector $X_x \in T_xM$ is

$$\widetilde{\omega}(X_x) = \iota_{X_x}\omega_x = \omega_x(X_x, \cdot),$$

where the "$\cdot$" indicates a spot waiting for someone to plug in a tangent vector.

Definition 3.3. Let $f\colon M \to \mathbb{R}$ be a smooth function. The **Hamiltonian vector field** of f, (sometimes called the **symplectic gradient** of f), is the smooth vector field X_f corresponding to the differential 1-form df via the bundle map

$$\widetilde{\omega}^{-1}\colon T^*M \to TM.$$

Note that X_f is the unique vector field on M such that

$$\iota_{X_f}\omega = df.$$

An arbitrary vector field $X \in \mathrm{Vec}(M)$ is called **Hamiltonian** if it is the Hamiltonian vector field of some smooth function $f\colon M \to \mathbb{R}$: $X = X_f$.

Proposition 3.4. *If $X \in \mathrm{Vec}(M)$ is Hamiltonian, then it is symplectic.*

Proof. If X is Hamiltonian, then there is some smooth function $f\colon M \to \mathbb{R}$ such that $X = X_f$. Then by the definition of X_f we have

$$d\iota_X\omega = d\iota_{X_f}\omega = d(df) = 0,$$

since $d^2 = 0$. Hence X is symplectic, by Proposition 3.2. □

4. Symplectic and Hamiltonian group actions

Let G be a Lie group, let (M, ω) be a symplectic manifold, and let $\mathcal{A}$ be a smooth action of G on M.

Definition 4.1. The action $\mathcal{A}$ of G on M is **symplectic** (with respect to ω) if each diffeomorphism $\mathcal{A}_g \colon M \to M$ is a symplectomorphism.

Let $\xi \in \mathfrak{g}$. Since the time t flow of the vector field ξ_M on M is the diffeomorphism $\mathcal{A}_{\exp(t\xi)} \colon M \to M$, if $\mathcal{A}$ is a symplectic action, then ξ_M is a symplectic vector field.

If each fundamental vector field ξ_M, $\xi \in \mathfrak{g}$, is symplectic, and additionally if $\exp \mathfrak{g} = G$, then the action $\mathcal{A}$ is symplectic.

Note 4.2. We use the notation $\langle \cdot, \cdot \rangle \colon \mathfrak{g}^* \to \mathfrak{g}\mathbb{R}$ to denote the natural pairing between covectors and vectors. To whit, if $\lambda \in \mathfrak{g}^*$ and $\xi \in \mathfrak{g}$, then

$$\langle \lambda, \xi \rangle := \lambda(\xi).$$

Although it doesn't seem like it, in some situations this notation makes things simpler.

Recall that the conjugation action of G on itself induces a linear action of G on $\mathfrak{g} = T_1 G$, denoted $\mathrm{Ad} \colon G \to \mathrm{Aut}(\mathfrak{g})$. This induces a linear action of G on the dual space $\mathfrak{g}^*$, denoted $\mathrm{Coad} \colon G \to \mathrm{Aut}(\mathfrak{g}^*)$ and defined by

$$\mathrm{Coad}_g(\lambda) := \lambda \circ \mathrm{Ad}_{g^{-1}}$$

for all $g \in G$ and $\lambda \in \mathfrak{g}^*$.

Definition 4.3. Let $\Phi \colon M \to \mathfrak{g}^*$ be a smooth map. For each $\xi \in \mathfrak{g}$, denote by $\Phi^\xi = \langle \Phi, \xi \rangle$ the smooth map $M \to \mathbb{R}$ defined by $x \mapsto \langle \Phi(x), \xi \rangle$. The map Φ is called a **moment map** for the action $\mathcal{A}$ of G on (M, ω) if it satisfies the following two properties.

(1) The map Φ is G-**equivariant**, meaning that for all $g \in G$ the diagram

$$\begin{array}{ccc} M & \xrightarrow{\Phi} & \mathfrak{g}^* \\ {\scriptstyle \mathcal{A}_g}\downarrow & & \downarrow{\scriptstyle \mathrm{Coad}_g} \\ M & \xrightarrow[\Phi]{} & \mathfrak{g}^* \end{array}$$

commutes.

(2) For all $\xi \in \mathfrak{g}$, the vector field ξ_M is the Hamiltonian vector field corresponding to the function $\Phi^{\xi}\colon M \to \mathbb{R}$, meaning that

$$d\langle \Phi, \xi \rangle = \omega\left(\xi_M, \cdot\right).$$

The action $\mathcal{A}$ of G on (M, ω) is called **Hamiltonian** if it is symplectic and there exists a moment map $\Phi\colon M \to \mathfrak{g}^*$ for it.

References

[G] T. Goldberg, *The symplectic integrability condition*, www.math.cornell.edu/~goldberg, 2008.