THE SYMPLECTIC INTEGRABILITY CONDITION

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CONTENTS

1. Introduction 1
2. Almost complex and complex structures 2
3. Almost symplectic and symplectic structures 3
4. Summary 4

1. INTRODUCTION

A **symplectic structure** on a manifold M is a differential 2-form ω satisfying two conditions:

(1) ω is **non-degenerate**, i.e. for each $p \in M$ and tangent vector $\vec{u}$ based at p, if $\omega_p(\vec{u}, \vec{v}) = 0$ for all tangent vectors $\vec{v}$ based at p, then $\vec{u}$ is the zero vector;
(2) ω is **closed**, i.e. the exterior derivative of ω is zero, i.e. $d\omega = 0$.

In trying to come up with answers to questions like "what do you do?" and "what is symplectic geometry?" that would be accessible to an advanced undergraduate or beginning graduate student, I've tried to come up with fairly intuitive descriptions of what the two symplectic structure conditions really mean.

Non-degeneracy is pretty easy, because my intended audience is certainly familiar with the dot product in Euclidean space, and probably familiar with more general machinery like inner products and bilinear forms. A **bilinear form** on a vector space over a field $\mathbb{F}$ is just an assignment of a number in $\mathbb{F}$ to each pair of

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vectors, in such a way that the assignment is linear in each vector in the pair. A bilinear form is called **non-degenerate** if the only thing that pairs to zero with every single vector is the zero vector. A 2-form on M is a collection $\omega = \{\omega_p \mid p \in M\}$ of skew-symmetric bilinear forms, one for each tangent space of M. Saying that ω is non-degenerate is saying that each of these bilinear forms is non-degenerate.

It's much less clear how to describe to the uninitiated what the closed condition means. It's even a bit unclear why this condition is required in the first place. A pretty nice answer came up yesterday, in a reading group I attend that is trying to learn about generalized complex structure. We are going through the PhD thesis of Marco Gualtieri, titled "Generalized Complex Geometry". It is available at the following websites:

`http://front.math.ucdavis.edu/0401.5221`
and
`http://front.math.ucdavis.edu/0703.5298`.

This was the first meeting, and Tomoo Matsumura,

`http://www.math.cornell.edu/People/Faculty/matsumura.html`,

was the speaker. He suggested that the requirement that $d\omega = 0$ is an **integrability condition**. I had never thought of it this way, but I probably will from now on.

2. Almost complex and complex structures

Let me first describe what integrability means for an almost complex structure on a manifold. A **complex structure on a vector space** V, where V is real and finite-dimensional, is a linear endomorphism $J\colon V \to V$ such that $J^2 = -\mathrm{id}_V$. Taking the determinant of both sides, we have $(\det J)^2 = (-1)^{\dim V}$. Since $(\det J)^2 \geq 0$, we must have $(-1)^{\dim V} = 1$, so $\dim V$ must be even. Furthermore, since $(\det J)^2 = 1$, we know $\det J \neq 0$, so J is a linear automorphism. The complex structure J makes V into a complex vector space, by setting

$$(a + b\sqrt{-1})\vec{v} := a\vec{v} + b\, J(\vec{v})$$

for $a, b \in \mathbb{R}$ and $\vec{v} \in V$.

The standard example is $\mathbb{R}^{2n}$ with its usual ordered basis, labelled $x_1, y_1, \ldots, x_n, y_n$, and complex structure J_0 defined by $J_0(x_j) = y_j$ and $J_0(y_j) = -x_j$ for all j. Putting $z_j = x_j + \sqrt{-1}y_j$, we obtain the usual ordered basis for $\mathbb{C}^n$ with its usual complex structure.

Let M be a $(2n)$-dimensional manifold. An **almost complex structure** on M is a smoothly-varying collection $J = \{J_p \mid p \in M\}$ of complex structures, one for each tangent space of M. (The existence of an almost complex structure forces $\dim M$ to be even.) An almost complex structure on M is just a bunch of complex structures on the tangent spaces glued together smoothly along M. Recall that as a manifold, all tangent spaces to a vector space can be canonically identified with the vector space itself, so a choice of complex structure on the vector space induces an almost complex structure on the vector space as a manifold.

An almost complex structure J on M is called a **complex structure** if the complex structures on the vector spaces fit together in an even nicer way. We require that there be a covering of M by coordinate neighborhoods such that on each such neighborhood J is the pullback of the standard complex structure on $\mathbb{R}^{2n}$. We require also that all transition maps for these coordinate charts be holomorphic with respect to the standard complex structure. This collection of coordinate charts form a complex atlas for M, and give M the structure of a complex manifold. (Notice that's it's easy to choose coordinates so that a single J_p looks like the standard one, J_0. We require that this hold not just at a single point, but in an entire neighborhood of the point.)

An almost complex structure M is called **integrable** if it is actually a complex structure. There are many integrability conditions for almost complex structures, such as the vanishing of the **Nijenhuis tensor** associated to an almost complex structure.

3. Almost symplectic and symplectic structures

Now we give a parallel discussion for symplectic structures. A **symplectic structure on a vector space** V, where V is real and finite-dimensional, is non-degenerate and skew-symmetric bilinear form $\Omega \colon V \times V \to \mathbb{R}$. Choose a basis for V and represent Ω by a matrix A relative to this basis. Because Ω is non-degenerate we know $\det A \neq 0$, and because it is skew-symmetric we know that $A = -A^{tr}$, so $\det A = \det(-A^{tr}) = (-1)^{\dim V} \det A^{tr} = (-1)^{\dim V} \det A$. Hence $(-1)^{\dim V} = 1$, so $\dim V$ must be even.

The standard example is $\mathbb{R}^{2n}$ with its usual ordered basis, labelled $x_1, y_1, \ldots, x_n, y_n$, and symplectic structure Ω_0 defined by $\Omega_0(x_i, y_j) = \delta_{ij}$ and $\Omega_0(x_i, x_j) = \Omega_0(y_i, y_j) = 0$, where δ_{ij} is the Kronecker delta.

Let M be a $(2n)$-dimensional manifold. An **almost symplectic structure** on M is a smoothly-varying collection $\omega = \{\omega_p \mid p \in M\}$ of symplectic structures, one for each tangent space of M. (The existence of an almost symplectic structure forces $\dim M$ to be even.) An almost symplectic structure on M is just a bunch of symplectic structures on the tangent spaces glued together smoothly along M.As before, a choice of symplectic structure on a vector space induces an almost symplectic structure on the vector space as a manifold.

An almost symplectic structure ω on M is called a **symplectic structure** if the symplectic structures on the vector spaces fit together in an even nicer way. Analogous to the complex structure case, we require that there be a covering of M by coordinate neighborhoods such that on each such neighborhood ω is the pullback of the standard complex structure on $\mathbb{R}^{2n}$. We require also that all transition maps for these coordinate charts be symplectic with respect to the standard symplectic structure. A manifold with a symplectic structure is a symplectic manifold.

Not much seems to be said about almost symplectic structures on manifolds, and so even less is said about **integrable** almost symplectic structures. But if one were to say something about them, surely the first thing would be to notice that, by Darboux's Theorem, there is an extremely simple integrability condition. This is exactly that ω be closed, i.e. that $d\omega = 0$.

4. Summary

To summarize, every manifold is locally isomorphic to some $\mathbb{R}^n$. An almost complex manifold is one equipped with a smoothly varying collection of complex structures on its tangent spaces. An almost complex manifold is a complex manifold if it is locally isomorphic to some $\mathbb{R}^{2n}$ with its standard complex structure. In this case, the almost complex structure is called integrable. Every previous sentence in this paragraph holds with the word "complex" replaced with "symplectic". There are many well-known conditions for an almost complex structure to be integrable. To the best of my knowledge, there is really only one well-known condition for an almost symplectic structure to be integrable, and this is the innocuous looking requirement that $d\omega = 0$.

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