

# Combinatorial Laplacians of Simplicial Complexes

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# Abstract

In this paper, we study the combinatorial Laplacian operator on the vector space of oriented chains over  $\mathbb{R}$  of a finite simplicial complex. We develop an easy method of computing the matrix of this operator from the adjacencies of simplices in the simplicial complex, and then apply this and results from linear algebra and simplicial homology to study properties of the Laplacian operator and its spectrum. We examine and explore connections between the combinatorial structure of simplicial complexes and their Laplacian spectra. Specific examples studied include certain classes of graphs and higher dimensional simplicial complexes, in particular cones of simplicial complexes, especially simplicial cones of dimension 2.

# Dedication

To my parents, for supporting me on whatever path I take,

and

To Paul, Grandma, Judy, and Grandmom, who probably would not have understood a word of this, but who would have loved it anyway.

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Let me warn you, I am not known for my brevity.

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# 1

## Introduction

The main purpose of this paper is to study the connections in the properties of finite simplicial complexes and the spectra of their Laplacian operators. This Laplacian operator is a generalization of a relatively well-studied Laplacian operator from graph theory, which in turn is, to a certain extent, a discrete version of the differential Laplacian operator. For some history behind the development of Laplacian operators for graphs and simplicial complexes, see the introduction of [DURE].

Our Laplacian for simplicial complexes is called a Combinatorial Laplacian, although for brevity's sake we usually drop the initial adjective, because it is a combinatorial invariant. The Laplacian operator and its spectrum do not depend on the geometry of the underlying simplicial complex, but instead in some way on how the various simplices in the complex are connected to each other. Perhaps unfortunately, our Laplacian and its spectrum are in no way topological invariants. The author has looked at literally dozens of 2-dimensional simplicial complexes that are all homeomorphic to a closed disk, but the Laplacian operator and spectra of any two of these examples were always quite different.

In Section 2.1, we state and prove many results about the theory of finitely-generated free abelian groups. These results are not actually used in the rest of the paper, but the work was done to prove that our work on Laplacians of simplicial complexes could essentially be done with oriented chains over  $\mathbb{Z}$  just as well as over  $\mathbb{R}$ , even though we work over  $\mathbb{R}$  throughout the rest of the paper. Section 2.2 presents many known and important results from linear algebra about eigenvalues and eigenvectors that will be used extensively later.

In Chapter 3, we introduce the graph theory Laplacian, and then develop some basic definitions concerning simplicial complexes before defining the Laplacian operator for simplicial complexes in Section 3.3. In Section 3.4 we rework our definitions and results from the rest of the section to develop reduced Laplacians of simplicial complexes, analogous to reduced simplicial homology.

In Chapter 4 we prove many extremely useful facts about the spectrum of the Laplacian operator, as well as the spectra of the family of operators closely related to the Laplacian. Finally, in Chapter 5 our work culminates in its application to specific families of and structures found within simplicial complexes. Section 5.1 presents results about the 1-skeletons of simplicial complexes, which are essentially the graphs living within all simplicial complexes. In this section we also characterize the Laplacian spectra of two major classes of graphs, complete graphs and bipartite graphs.

Section 5.1 contains what is probably the most extensive and intense work of this project, on cones of simplicial complexes of any dimension and simplicial cones of dimension 2 or less. The major theorems contained in this section are then used to characterize completely the Laplacian spectra of several families of simplicial cones, namely flapwheels, pinwheels, asterisks, and simplices themselves.

# 2

## Algebraic Preliminaries

### 2.1 Some Group Theory - Adjoint Homomorphisms

This section develops some definitions and results about homomorphisms between finitely generated free abelian groups. Background information on free abelian groups can be found in any text on abstract algebra, such as [FRA94, Section 4.4]. Most of the following definition comes from [MUN84, page 21].

**Definition.** Let  $G$  be a free abelian group with basis  $\{\alpha_1, \dots, \alpha_n\}$ , and let  $g \in G$ . Then  $g$  can be written uniquely as a finite sum

$$g = \sum_{i=1}^n k_i \alpha_i$$

for  $k_1, k_2, \dots, k_n \in \mathbb{Z}$ . The column vector  $(k_1 k_2 \dots k_n)^T$ , where the superscript  $T$  denotes the usual matrix transpose, is called the **coordinate vector** of  $g$  relative to the given basis for  $G$ . △

It is usually clear from the context whether we are referring to a group element or its coordinate vector, so we shall abuse notation and refer to both by the name of the element.

The fact that the coordinate vector representation defined above is unique follows from the uniqueness of an element's representation as the finite sum of basis elements. Also, it is easy to see that the coordinate vector of the sum of two elements is the sum of the coordinate vectors of those two elements. Finally, note that if the coordinate vectors of two elements of a finitely generated free abelian groups are identical, then the elements must be identical.

The following definition comes from [MUN84, page 55].

**Definition.** Let  $G$  and  $G'$  be free abelian groups with finite bases  $\{\alpha_1, \dots, \alpha_n\}$  and  $\{\beta_1, \dots, \beta_m\}$ , respectively. If  $f: G \rightarrow G'$  is a homomorphism, then for all  $j \in \{1, 2, \dots, n\}$

we have

$$f(\alpha_j) = \sum_{i=1}^m \lambda_{ij} \beta_i$$

for some unique integers  $\lambda_{ij}$ . The  $m \times n$  matrix whose  $ij$ th coordinate is given by  $\lambda_{ij}$  for all integers  $i$  and  $j$  with  $1 \leq i \leq m$  and  $1 \leq j \leq n$  is called the **matrix of  $f$**  relative to the given bases for  $G$  and  $G'$ .  $\triangle$

Let  $G$  be a free abelian group with finite basis  $\{\alpha_1, \dots, \alpha_n\}$ . For all  $x, y \in G$ , let  $\langle x, y \rangle$  denote the standard dot product for vectors, as given in [FIS97, Chapter 6], performed on the coordinate vectors of  $x$  and  $y$ . For all elements  $x, y \in G$ , since their coordinate vectors have integer entries, we see that  $\langle x, y \rangle$  must be an integer. Also, for all integers  $i$  and  $j$  with  $1 \leq i, j \leq n$ , we see that  $\langle \alpha_i, \alpha_j \rangle$  is 1 if  $i = j$  and 0 if  $i \neq j$ . In this sense, this basis for  $G$  is in some way similar to an orthonormal basis of a vector space.

The following results and proofs are modeled after those in [FIS97, Section 6.3]. The goal of these theorems is to develop a notion of an adjoint homomorphism, similar to the idea of an adjoint linear operator in linear algebra.

**Lemma 2.1.1.** *Let  $G$  be a free abelian group with basis  $\{\alpha_1, \dots, \alpha_n\}$ . Let  $y, z \in G$ . If  $\langle x, y \rangle = \langle x, z \rangle$  for all  $x \in G$ , then  $y = z$ .*

**Proof.** Suppose  $\langle x, y \rangle = \langle x, z \rangle$  for all  $x \in G$ . Then for  $x = y$  we obtain  $\langle y, y \rangle = \langle y, z \rangle$ , and for  $x = z$  we obtain  $\langle z, y \rangle = \langle z, z \rangle$ . Since the coordinate vectors of  $y$  and  $z$  are both real, we know that the dot product commutes here, so  $\langle y, z \rangle = \langle z, y \rangle$ . Therefore, subtracting the equation  $\langle z, z \rangle = \langle z, y \rangle$  from the equation  $\langle y, y \rangle = \langle y, z \rangle$ , we see that  $\langle y - z, y - z \rangle = \langle y, y \rangle - \langle z, z \rangle = 0$ . From [FIS97, Theorem 6.1], we know that  $\langle y - z, y - z \rangle = 0$  implies that  $y - z = \vec{0}$ , so  $y = z$ .  $\square$

**Theorem 2.1.2.** *Let  $G$  be a free abelian group with basis  $\{\alpha_1, \dots, \alpha_n\}$ , and let  $h: G \rightarrow \mathbb{Z}$  be a homomorphism. Then there exists a unique  $y \in G$  such that  $h(x) = \langle x, y \rangle$  for all  $x \in G$ .*

**Proof.** Let  $y = \sum_{i=1}^n h(\alpha_i) \alpha_i$ , and let  $f: G \rightarrow \mathbb{Z}$  be the map given by  $f(x) = \langle x, y \rangle$  for all  $x \in G$ . Let  $a, b \in G$ . By standard properties of the dot product of vectors, we have

$$f(a + b) = \langle (a + b), y \rangle = \langle a, y \rangle + \langle b, y \rangle = f(a) + f(b),$$

so  $f$  is a homomorphism.

Let  $j \in \{1, 2, \dots, n\}$ . Then  $f(\alpha_j) = \langle \alpha_j, y \rangle = \langle \alpha_j, \sum_{i=1}^n h(\alpha_i) \alpha_i \rangle = \sum_{i=1}^n h(\alpha_i) \langle \alpha_j, \alpha_i \rangle$ . We know that summands of this last sum are 0 unless  $i = j$ , in which case the dot product in the sum is 1, so the last sum reduces to  $h(\alpha_j)$ . Since  $f$  and  $h$  agree on all basis elements of  $G$ , it follows that  $f = h$ .

To show that  $y$  is unique, suppose  $h(x) = \langle x, y' \rangle$  for all  $x \in G$ , for some  $y' \in G$ . Then  $\langle x, y \rangle = \langle x, y' \rangle$  for all  $x \in G$ , so it follows from Lemma 2.1.1 that  $y = y'$ .  $\square$

**Theorem 2.1.3.** *Let  $G$  and  $G'$  be free abelian groups with finite bases  $\{\alpha_1, \dots, \alpha_n\}$  and  $\{\beta_1, \dots, \beta_m\}$ , respectively, and let  $H: G \rightarrow G'$  be a homomorphism. Then there exists a unique homomorphism  $H^*: G' \rightarrow G$  such that for all  $x \in G$  and  $y \in G'$  we have*

$$\langle H(x), y \rangle = \langle x, H^*(y) \rangle.$$

**Proof.** Let  $y \in G'$ . Let  $g: G \rightarrow \mathbb{Z}$  be the map given by  $g(x) = \langle H(x), y \rangle$  for all  $x \in G$ . We will show that  $g$  is a homomorphism. Let  $a, b \in G$ . Recalling properties of the dot product and that  $H$  is a homomorphism, we have

$$g(a + b) = \langle H(a + b), y \rangle = \langle H(a) + H(b), y \rangle = \langle H(a), y \rangle + \langle H(b), y \rangle = g(a) + g(b).$$

By Theorem 2.1.2 we know there is a unique  $q \in G$  such that  $g(x) = \langle x, q \rangle$  for all  $x \in G$ ; that is,  $\langle H(x), y \rangle = \langle x, q \rangle$  for all  $x \in G$ . We define a map  $H^*: G' \rightarrow G$  on the element  $y \in G'$  by  $H^*(y) = q$ . Since  $y$  was chosen arbitrarily, this process defines the map  $H^*$  on every element in  $G'$ . We see that this map has the property that for all  $x \in G$  and  $y \in G'$  we have  $\langle H(x), y \rangle = \langle x, H^*(y) \rangle$ . We will now show that  $H^*$  is a homomorphism.

Let  $c, d \in G'$ . For all  $x \in G$ , we have  $\langle x, H^*(c + d) \rangle = \langle H(x), c + d \rangle = \langle H(x), c \rangle + \langle H(x), d \rangle = \langle x, H^*(c) \rangle + \langle x, H^*(d) \rangle = \langle x, H^*(c) + H^*(d) \rangle$ . Since  $x$  is arbitrary, by Lemma 2.1.1 we have  $H^*(c + d) = H^*(c) + H^*(d)$ .

To show that  $H^*$  is unique, suppose  $U: G' \rightarrow G$  is a homomorphism such that for all  $x \in G$  and  $y \in G'$  we have  $\langle H(x), y \rangle = \langle x, U(y) \rangle$ . Then  $\langle x, H^*(y) \rangle = \langle H(x), y \rangle = \langle x, U(y) \rangle$  for all  $x \in G$  and  $y \in G'$ , so  $H^*(y) = U(y)$  for all  $y \in G'$ , so  $H^* = U$ .  $\square$

**Definition.** The homomorphism  $H^*$  defined in the above result, under the conditions given in the statement of the theorem, is called the **adjoint homomorphism** of  $H$ .  $\triangle$

**Lemma 2.1.4.** *Let  $G$  be a free abelian group with finite basis  $A = \{\alpha_1, \dots, \alpha_n\}$ , and let  $y \in G$ . Then*

$$y = \sum_{i=1}^n \langle y, \alpha_i \rangle \alpha_i.$$

**Proof.** Let  $y = \sum_{i=1}^n a_i \alpha_i$  be the unique representation of  $y$  with respect to the basis  $A$ , where  $a_1, a_2, \dots, a_n \in \mathbb{Z}$ . Let  $j \in \{1, 2, \dots, n\}$ . Then  $\langle y, \alpha_j \rangle = \langle \sum_{i=1}^n a_i \alpha_i, \alpha_j \rangle = \sum_{i=1}^n a_i \langle \alpha_i, \alpha_j \rangle$ . The dot product in this last sum is 0 if  $i \neq j$  and 1 if  $i = j$ , so this sum reduces to  $a_j \langle \alpha_j, \alpha_j \rangle = a_j$ . The lemma follows by replacing the coefficient  $a_i$  with  $\langle y, \alpha_i \rangle$  for all  $i \in \{1, 2, \dots, n\}$  in the unique representation of  $y$  with respect to  $A$  given at the beginning of this proof.  $\square$

**Lemma 2.1.5.** *Let  $G$  and  $G'$  be free abelian groups with finite bases  $A = \{\alpha_1, \dots, \alpha_n\}$  and  $B = \{\beta_1, \dots, \beta_m\}$ , respectfully, and let  $H: G \rightarrow G'$  be a homomorphism. Let  $[H]$  be the matrix of  $H$  with respect to the bases  $A$  and  $B$ . Then for all for all integers  $i$  and  $j$  with  $1 \leq i \leq m$  and  $1 \leq j \leq n$  we have*

$$[H]_{ij} = \langle H(\alpha_j), \beta_i \rangle.$$

**Proof.** Let  $j \in \{1, 2, \dots, n\}$ . By Lemma 2.1.4, we know that  $H(\alpha_j) = \sum_{i=1}^m \langle H(\alpha_j), \beta_i \rangle \beta_i$ . By the definition of the matrix of a homomorphism, we see that the  $ij$ th entry of  $[H]$  is the coefficient of  $\beta_i$  in this sum for  $H(\alpha_j)$ . It follows that  $[H]_{ij} = \langle H(\alpha_j), \beta_i \rangle$  for all integers  $i$  and  $j$  with  $1 \leq i \leq m$  and  $1 \leq j \leq n$ .  $\square$

**Theorem 2.1.6.** *Let  $G$  and  $G'$  be free abelian groups with finite bases  $A = \{\alpha_1, \dots, \alpha_n\}$  and  $B = \{\beta_1, \dots, \beta_m\}$ , respectfully, and let  $H: G \rightarrow G'$  be a homomorphism. Let  $[H]$*

denote the matrix of  $H$  with respect to the bases  $A$  and  $B$ , and let  $[H^*]$  be the matrix of  $H^*$  with respect to these bases. Then

$$[H^*] = [H]^T.$$

**Proof.** Let  $i$  and  $j$  be integers with  $1 \leq i \leq m$  and  $1 \leq j \leq n$ . Using Lemma 2.1.5 and the fact that the dot product of vectors with real entries is commutative, we have

$$[H^*]_{ij} = \langle H^*(\beta_j), \alpha_i \rangle = \langle \alpha_i, H^*(\beta_j) \rangle = \langle H(\alpha_i), \beta_j \rangle = [H]_{ji}.$$

□

## 2.2 Some Linear Algebra - Eigenvalues and Eigenvectors

The purpose of this section is to build up a number of important tools from linear algebra that we will have call to use in later sections.

**Definition.** A **multiset** is a pair  $M = (A, m)$ , where  $A$  is a set and  $m$  is a function  $m: A \rightarrow \mathbb{N}$ , where  $\mathbb{N}$  denotes the nonnegative integers. We think of the multiset  $M$  as containing each element  $a \in A$  a total of  $m(a)$  times. The function  $m$  is the **multiplicity function**, and for each  $a \in A$ , the **multiplicity** of  $a$  is  $m(a)$ . (In practice, we usually do not explicitly mention the multiplicity function of a multiset.)

Given two finite multisets  $X$  and  $Y$ , we define the **multiset union** of  $X$  and  $Y$ , denoted  $X \cup_M Y$ , to be the multiset containing exactly the elements of  $X$  and  $Y$  with the multiplicity of an element in the multiset union given by the sum of that element's multiplicities in  $X$  and  $Y$ .

Let  $X$  be a multiset. We let  $(X)_{NZ}$  be the multiset that is identical to  $X$  except that  $(X)_{NZ}$  does not contain 0. For all elements  $x$  and nonnegative integers  $i$ , we let  $[x]^i$  denote the element  $x$  with multiplicity  $i$ . Suppose the size of  $X$  is  $n \in \mathbb{Z}^+$ . For all integers  $m \geq n$ , we define the multiset  $\binom{X}{m} = X \cup_M \{[0]^{m-n}\}$ , and call it the **scaled multiset** of  $X$  scaled to size  $m$ .

If  $X$  and  $Y$  are two finite, ordered multisets of equal size, we define the **multiset sum** of  $X$  and  $Y$ , denoted  $X +_M Y$ , to be the ordered multiset with the same size as  $X$  and  $Y$  whose elements are the component-wise sums of the elements of  $X$  and  $Y$ . (If one of the multisets in the multiset sum consists of a single element with some multiplicity, then we can relax the condition that the multisets be ordered, since in that case there is no ambiguity.)

△

Let  $V, W$  be finite dimensional vector spaces over a field  $F$ , and let  $T: V \rightarrow V$  be a linear operator. The operator  $T$  is called **diagonalizable** if there exists a basis  $\beta$  for  $V$  such that the matrix of  $T$  relative to  $\beta$ , denoted  $[T]_\beta$ , is a diagonal matrix. (The problem of determining whether or not  $T$  is diagonalizable reduces to the problem of finding a basis for  $V$  consisting of eigenvectors of  $T$ .)

The multiset of eigenvalues of  $T$  is the **spectrum** of  $T$ , denoted  $Spec(T)$ , and the multiset of nonzero eigenvalues of  $T$  is denoted  $Spec_{NZ}(T)$ . For each eigenvalue  $\lambda \in F$  of  $T$ ,

we let  $E_\lambda(T)$  denote the corresponding eigenspace. (If there is no confusion, sometimes we drop the  $(T)$ .) The zero eigenspace of  $T$ , also known as the null space of  $T$ , is denoted  $N(T)$ , and the union of the eigenspaces of  $T$  associated with nonzero eigenvalues is  $E_{NZ}(T)$ . The zero operator from  $V$  to  $W$  is written  $\mathbf{0}_{V,W}$ , and the zero operator from  $V$  to  $V$  is abbreviated  $\mathbf{0}_V$ .

Definitions for all terms used in the rest of this section can be found in any text on linear algebra, such as [FIS97]. We will now prove several basic results about the eigenvalues and eigenvectors of certain types linear operators.

The following result is stated in [FIS97, Exercise 15, page 356], although the proof given here is ours.

**Theorem 2.2.1 (Simultaneous Diagonalization).** *Let  $V$  be a finite dimensional inner product space over a field  $F$ , and suppose  $T$  and  $U$  are self-adjoint linear operators on  $V$  such that  $TU = UT$ . Then there exists a basis for  $V$  whose elements are eigenvectors of both  $T$  and  $U$ .*

**Proof.** Let  $\lambda_1, \dots, \lambda_k \in F$  be the distinct eigenvalues of  $T$ . Let  $i \in \{1, \dots, k\}$ , and let  $W = E_{\lambda_i}(T)$  be the eigenspace of  $T$  associated with the eigenvalue  $\lambda_i$ . We see immediately that  $W$  is  $T$ -invariant (meaning that  $T(W) \subseteq W$ ). In fact  $W$  is also  $U$ -invariant. Let  $v \in W$ . Then

$$T(U(v)) = U(T(v)) = U(\lambda_i v) = \lambda_i U(v),$$

so  $U(v)$  is an eigenvector of  $T$  associated with  $\lambda_i$ , so  $U(W) \subseteq W$ .

By [FIS97, Theorem 6.17] we know there is a basis  $\{w_1, \dots, w_n\}$  for  $V$  consisting of eigenvectors of  $U$ , implying that  $U$  is diagonalizable. The same is true of  $T$ , so from this and [FIS97, Theorem 5.16] we have that

$$V = \bigoplus_{i=1}^k E_{\lambda_i}(T).$$

Let  $j \in \{1, \dots, n\}$ , and let  $\alpha_j \in F$  denote the eigenvalue of  $U$  with which the basis element  $w_j$  is associated. By the definition of the direct sum, we know there is a representation of  $w_j$  as the sum  $w_j = v_{j1} + v_{j2} + \dots + v_{jk}$ , where  $v_{ji} \in E_{\lambda_i}(T)$  for all  $i \in \{1, \dots, k\}$ . Then

$$U(w_j) = \alpha_j w_j = \alpha_j v_{j1} + \alpha_j v_{j2} + \dots + \alpha_j v_{jk} \tag{2.2.1}$$

and

$$U(w_j) = U(v_{j1} + v_{j2} + \dots + v_{jk}) = U(v_{j1}) + U(v_{j2}) + \dots + U(v_{jk}). \tag{2.2.2}$$

Note that since every eigenspace of  $T$  is  $U$ -invariant we have  $U(v_{ji}) \in E_{\lambda_i}(T)$ , and of course also  $\alpha_j v_{ji} \in E_{\lambda_i}(T)$ , for all  $i \in \{1, \dots, k\}$ . We see then that Equations 2.2.1 and 2.2.2 are sum representations of  $U(w_j)$  with respect to the direct sum decomposition of  $V$  into eigenspaces of  $T$ . By [FIS97, Theorem 5.15c] we know that such representations are unique, so it must be that  $\alpha_j v_{ij} = U(v_{ij})$ , and so in fact  $v_{ij}$  is an eigenvector of  $U$  as well as of  $T$ , for all  $i \in \{1, \dots, k\}$ .

Since the above arguments hold for arbitrary  $j \in \{1, \dots, n\}$ , we see that every element of  $B = \{v_{11}, v_{12}, \dots, v_{1k}, v_{21}, v_{22}, \dots, v_{2k}, \dots, v_{n1}, v_{n2}, \dots, v_{nk}\}$  is an eigenvector of both  $T$

and  $U$ . Furthermore, because  $w_j = v_{j1} + v_{j2} + \dots + v_{jk}$  for all  $j \in \{1, \dots, n\}$  and the set  $\{w_1, \dots, w_n\}$  is a basis for  $V$ , it must be that the set  $B$  spans  $V$ . Therefore it follows from [FIS97, Theorem 1.9] that some subset of  $B$  is a basis for  $V$ , and we see that the vectors of this basis are eigenvectors of both  $T$  and  $U$ .  $\square$

**Lemma 2.2.2.** *Let  $V$  be a finite dimensional vector space over a field  $F$ , and let  $W_1, W_2 \subseteq V$  be subspaces. Suppose  $B = \{x_1, x_2, \dots, x_n\}$  is a basis for  $V$  such that  $B_1 = \{x_1, x_2, \dots, x_i\}$  and  $B_2 = \{x_j, x_{j+1}, \dots, x_n\}$  are bases for  $W_1$  and  $W_2$ , respectively, for some  $i, j \in \{1, 2, \dots, n\}$ . Then*

$$B_1 \cap B_2$$

*is a basis for the subspace  $W_1 \cap W_2$ .*

**Proof.** We know the intersection of two linearly independent sets is linearly independent, so we must show that the intersection basis spans the intersection subspace.

If  $W_1 \cap W_2 = \{0\}$ , then their bases must be disjoint or else one of the basis elements would be contained in the intersection. In this case the lemma's desired result is certainly satisfied. Suppose there is some nonzero  $v \in W_1 \cap W_2$ . Then  $v \in W_1$  and  $v \in W_2$ , so we have

$$a_1x_1 + a_2x_2 + \dots + a_ix_i = v = b_jx_j + b_{j+1}x_{j+1} + \dots + b_nx_n$$

for some scalars  $a_1, a_2, \dots, a_i, b_j, b_{j+1}, \dots, b_n$ .

First we must show that  $B_1$  and  $B_2$  are not disjoint. Suppose  $B_1$  and  $B_2$  are disjoint. Then  $i < j$ , so from the above equations we have

$$a_1x_1 + \dots + a_ix_i + 0 \cdot x_{i+1} + \dots + 0 \cdot x_{j-1} - b_jx_j - \dots - b_nx_n = 0.$$

Since  $B$  is linearly independent, this implies that  $a_1 = \dots = a_i = b_j = \dots = b_n = 0$ , which means that  $v = 0$ , a contradiction. Therefore  $B_1$  and  $B_2$  are not disjoint, so  $j \leq i$ .

This means that we can write

$$a_1x_1 + \dots + a_jx_j + \dots + a_ix_i = v = b_jx_j + \dots + b_ix_i + \dots + b_nx_n,$$

and so

$$a_1x_1 + \dots + a_{j-1}x_{j-1} + (a_j - b_j)x_j + \dots + (a_i - b_i)x_i - b_{i+1}x_{i+1} - \dots - b_nx_n = 0.$$

Since  $B$  is linearly independent, this implies that  $a_1 = \dots = a_{j-1} = (a_j - b_j) = \dots = (a_i - b_i) = b_{i+1} = \dots = b_n = 0$ , so in fact  $v = a_ix_i + \dots + a_jx_j = b_ix_i + \dots + b_jx_j$ . Therefore  $v \in \text{span}\{x_i, \dots, x_j\} = \text{span}(B_1 \cap B_2)$ .  $\square$

**Lemma 2.2.3.** *Let  $V$  be a finite dimensional vector space over a field  $F$ , and let  $T$  be an operator on  $V$ . Suppose  $T$  is diagonalizable, and let  $\lambda_1, \dots, \lambda_k \in F$  denote the distinct eigenvalues of  $T$ . If  $B$  is a basis of eigenvectors of  $T$ , then there exists a partition  $\{B_1, \dots, B_k\}$  of  $B$  such that  $B_i$  is a basis for  $E_{\lambda_i}$  for all  $i \in \{1, \dots, k\}$ .*

**Proof.** Let  $B = \{x_1, \dots, x_n\}$  be a basis of eigenvectors of  $T$ , and for each  $j \in \{1, \dots, n\}$  let  $\lambda_j \in F$  denote the eigenvalue of  $T$  with which  $x_j$  is associated. We first show that there is an eigenvector in  $B$  for each eigenvalue of  $T$ . Let  $\lambda \in F$  be an eigenvalue of  $T$ .



Then there is some nonzero  $v \in V$  such that  $T(v) = \lambda v$ . Since  $B$  is a basis there exist unique  $a_1, \dots, a_n \in F$  such that  $v = a_1x_1 + \dots + a_nx_n$ , and since  $v \neq 0$  there must be some  $j \in \{1, \dots, n\}$  such that  $a_j \neq 0$ . Then

$$T(v) = \lambda v = \lambda a_1x_1 + \dots + \lambda a_nx_n$$

and

$$T(v) = T(a_1x_1 + \dots + a_nx_n) = a_1T(x_1) + \dots + a_nT(x_n) = a_1\lambda_1x_1 + \dots + a_n\lambda_nx_n.$$

Since  $B$  is a basis, we have  $\lambda a_j = \lambda_j a_j$ , and since  $a_j \neq 0$  we have  $\lambda = \lambda_j$ . Hence  $x_j \in B$  is an eigenvector associated with  $\lambda$ .

For each  $i \in \{1, \dots, k\}$  let  $B_i = E_{\lambda_i} \cap B$ . Since the intersection of distinct eigenspaces is  $\{0\}$ , we see that  $\{B_1, \dots, B_k\}$  is a partition of  $B$ . Since  $T$  is diagonalizable, [FIS97, Theorem 5.16] tells us that  $V$  is the direct sum of the eigenspaces of  $T$ . [FIS97, Theorem 5.15(d)] states that the union of bases for the subspaces of a direct sum forms a basis for the direct sum itself, and this implies that the sum of the dimensions of the subspaces of a direct sum is the dimension of the direct sum. Therefore

$$\sum_{i=1}^k \dim(E_{\lambda_i}) = \dim(V) = n.$$

For each  $i \in \{1, \dots, k\}$ , since  $B_i \subseteq E_{\lambda_i}$  and  $B_i$  is linearly independent, we have  $|B_i| \leq \dim(E_{\lambda_i})$ . Suppose there is some  $j \in \{1, \dots, k\}$  such that  $|B_j| < \dim(E_{\lambda_j})$ . Then  $n = |B| = |B_1| + \dots + |B_k| < \sum_{i=1}^k \dim(E_{\lambda_i}) = n$ , a contradiction. Therefore, it must be that  $|B_i| = \dim(E_{\lambda_i})$ , and hence  $B_i$  is a basis for  $E_{\lambda_i}$ , for all  $i \in \{1, \dots, k\}$ .  $\square$

**Lemma 2.2.4.** *Let  $V$  be a finite dimensional vector space over a field  $F$ , and let  $T$  and  $U$  be linear operators on  $V$  such that  $TU = \mathbf{0}_V = UT$ . Then  $E_{NZ}(T) \subseteq N(U)$  and  $E_{NZ}(U) \subseteq N(T)$ .*

**Proof.** Let  $x \in E_{NZ}(T)$ , and suppose  $\lambda \in F$  is the nonzero eigenvalue of  $T$  with which  $x$  is associated. Then  $\vec{0} = UT(x) = U(T(x)) = U(\lambda x) = \lambda U(x)$ . Since  $\lambda \neq 0$ , it must be that  $U(x) = \vec{0}$ , so  $x \in N(U)$ . Therefore  $E_{NZ}(T) \subseteq N(U)$ .

The exact same argument holds with the roles of  $T$  and  $U$  reversed, implying that  $E_{NZ}(U) \subseteq N(T)$ .  $\square$

The following is a very important result about pairs of operators on an inner product space that have a very particular relationship to each other.

**Theorem 2.2.5.** *Let  $V$  be a finite dimensional inner product space over a field  $F$ , and let  $T$  and  $U$  be self-adjoint linear operators on  $V$  such that  $TU = \mathbf{0}_V = UT$ . Then  $\text{Spec}_{NZ}(T+U) = \text{Spec}_{NZ}(T) \cup_M \text{Spec}_{NZ}(U)$  and  $N(T+U) = N(T) \cap N(U)$ .*

**Proof.** Since  $T$  and  $U$  commute, by Theorem 2.2.1 we know there is a basis  $B$  consisting of eigenvectors of both  $T$  and  $U$ . Let  $G = B \cap E_{NZ}(T)$ , the set of eigenvectors in  $B$  associated with nonzero eigenvalues of  $T$ ; let  $H = B \cap E_{NZ}(U)$ , the set of eigenvectors in  $B$  associated with nonzero eigenvalues of  $U$ ; and let  $J = N(T) \cap B \cap N(U)$ , the set of

eigenvectors in  $B$  associated with the eigenvalue 0 with respect to both  $T$  and  $U$ . We will demonstrate that  $\{G, H, J\}$  is a partition of the basis  $B$ .

By Lemma 2.2.4 we know that an eigenvector associated with a nonzero eigenvalue with respect to either  $T$  or  $U$  is in the nullspace of the other operator, so  $G$  and  $H$  are disjoint subsets. Naturally any element in the nullspaces of both  $T$  and  $U$  cannot be in either  $G$  or  $H$ , so  $J$  is disjoint from both  $G$  and  $H$ . We see that  $G, H, J$  are pairwise disjoint. Now we will show that  $G \cup H \cup J = B$ .

Let  $x \in B$ . We know that  $x$  is either in  $N(T)$  or in  $E_{NZ}(T)$ . If  $x \in N(T)$ , then either  $x \in N(U)$  or  $x \in E_{NZ}(U)$ , meaning that  $x \in J$  or  $x \in H$ , respectively. On the other hand, if  $x \in E_{NZ}(T)$  then we see that  $x \in G$ . Hence  $G \cup H \cup J = B$ .

Now we will show that every element of  $B$  is an eigenvector of  $T + U$ . First, note that by Lemma 2.2.4 we have  $G \subseteq N(U)$  and  $H \subseteq N(T)$ . Suppose  $x \in B$ . If  $x \in G$ , then  $(T + U)(x) = T(x) + U(x) = \lambda x + 0 = \lambda x$ , where  $\lambda \in F$  is the nonzero eigenvalue of  $T$  associated with  $x$ . If  $x \in H$ , then  $(T + U)(x) = T(x) + U(x) = 0 + \lambda' x = \lambda' x$ , where  $\lambda' \in F$  is the nonzero eigenvalue of  $U$  associated with  $x$ . Finally, if  $x \in J$ , then  $(T + U)(x) = T(x) + U(x) = 0 + 0 = 0$ . Since  $G \cup H \cup J = B$ , this implies that  $x$  is an eigenvector of  $T + U$ .

Let  $\lambda_1, \dots, \lambda_k \in F$  be the distinct nonzero eigenvalues associated with the elements of  $B$ , with respect to  $T + U$ . By Lemma 2.2.3, there is a partition  $\{B_1, \dots, B_k\}$  of  $B$  such that  $B_i$  is a basis for  $E_{\lambda_i}(T + U)$  for all  $i \in \{1, \dots, k\}$ . Let  $j \in \{1, \dots, k\}$ .

Suppose  $\lambda_j \neq 0$ . In showing that  $B$  is composed of eigenvectors of  $T + U$  we saw that this means that the elements in  $B_j$  are precisely the elements of  $B$  that are associated with the eigenvalue  $\lambda_j$  with respect to either  $T$  or  $U$ , but not both, because  $G$  and  $H$  are disjoint. Hence

$$\dim(E_{\lambda_j}(T)) + \dim(E_{\lambda_j}(U)) = |B_j| = \dim(E_{\lambda_j}(T + U)).$$

This means that the multiplicity of a nonzero eigenvalue in  $T + U$  is the sum of its multiplicities in  $T$  and in  $U$ . Since we chose the nonzero eigenvalue  $\lambda_j$  arbitrarily, it follows that

$$\text{Spec}_{NZ}(T + U) = \text{Spec}_{NZ}(T) \cup_M \text{Spec}_{NZ}(U).$$

Suppose  $\lambda_j = 0$ . As before, in showing that  $B$  is composed of eigenvectors of  $T + U$ , we saw that elements of  $B$  that are in the nullspace of  $T + U$  are precisely the elements of  $J$ , which consists of those elements which are in the intersection of the nullspaces of both  $T$  and  $U$  with the basis  $B$ . By Lemma 2.2.2, we see that  $J$  is a basis for the subspace  $N(T) \cap N(U)$  of  $V$ , so it must be that  $N(T + U) = \text{span}(B_j) = N(T) \cap N(U)$ .  $\square$

**Theorem 2.2.6.** *Let  $m$  and  $n$  be positive integers, let  $F$  be a field, and let  $U: F^m \rightarrow F^n$  and  $T: F^n \rightarrow F^m$  be linear operators. Then  $\text{Spec}_{NZ}(UT) = \text{Spec}_{NZ}(TU)$ .*

**Proof.** For all positive integers  $k$ , we let  $\vec{0}_k$  denote the zero column vector of dimension  $k$ .

Suppose  $\lambda \in F$  is a nonzero eigenvalue of  $UT$ . Let  $x \in F^m$  be an eigenvector of  $UT$  associated with  $\lambda$ . Then  $(UT)x = \lambda x$ , so  $(TU)Tx = T(UT)x = T\lambda x = \lambda(Tx)$ . If  $Tx = \vec{0}_n$ , then  $\lambda x = UTx = U(\vec{0}_n) = \vec{0}_m$ , and since  $x$  is an eigenvector and so must be nonzero, this implies that  $\lambda = 0$ , a contradiction. Therefore  $Tx \neq \vec{0}_n$ , so  $Tx$  is an eigenvector of

$TU$  associated with  $\lambda$ . A completely parallel argument to the one above shows that if  $y$  is an eigenvector of  $TU$  associated with some nonzero eigenvalue  $\lambda' \in F$ , then  $Ux$  is an eigenvector of  $UT$  associated with  $\lambda'$ .

From these two results, we conclude that a nonzero  $\lambda \in F$  is an eigenvalue of  $UT$  iff it is an eigenvalue of  $TU$ . We also have that  $U$  maps eigenvectors of  $TU$  to eigenvectors of  $UT$  associated with the same eigenvalue, and that  $T$  maps eigenvectors of  $UT$  to eigenvectors of  $TU$  associated with the same eigenvalue. Hence, for any nonzero eigenvalue  $\lambda \in F$  of  $UT$  and  $TU$ , we can define the following two functions.

Let  $\phi: E_\lambda(UT) \rightarrow E_\lambda(TU)$  and  $\psi: E_\lambda(TU) \rightarrow E_\lambda(UT)$  be given by

$$\phi(x) = Tx$$

and

$$\psi(y) = \frac{1}{\lambda}Uy$$

for all  $x \in E_\lambda(UT)$  and  $y \in E_\lambda(TU)$ . For all  $x \in E_\lambda(UT)$  and  $y \in E_\lambda(TU)$  we have

$$(\psi \cdot \phi)(x) = \frac{1}{\lambda}UTx = \frac{1}{\lambda}\lambda x = x$$

and

$$(\phi \cdot \psi)(y) = T\frac{1}{\lambda}Uy = \frac{1}{\lambda}TUY = \frac{1}{\lambda}\lambda y = y,$$

so  $\phi$  and  $\psi$  are inverses. Therefore  $E_\lambda(UT)$  and  $E_\lambda(TU)$  are isomorphic as subspaces, so in particular they must have the same dimension. Since the dimension of an eigenspace is the multiplicity of the eigenvalue with which that space is associated, it follows that the eigenvalue  $\lambda$  has the same multiplicity in  $UT$  and  $TU$ . Since this holds for all nonzero eigenvalues of  $UT$  and  $TU$ , it follows that  $\text{Spec}_{NZ}(UT) = \text{Spec}_{NZ}(TU)$ .  $\square$

Now we turn our attention to a property of linear operators on an inner product space that has a very important implication for the spectra of those operators.

**Definition.** Let  $V$  be a finite-dimensional vector space over  $\mathbb{R}$ , and let  $T: V \rightarrow V$  be a linear operator. We say  $T$  is **positive semidefinite** if  $T$  is self-adjoint and

$$\langle T(v), v \rangle \geq 0$$

for all vectors  $v \in V$ , where  $\langle \cdot, \cdot \rangle$  denotes the standard inner product over  $\mathbb{R}^n$ .  $\triangle$

**Lemma 2.2.7.** *Let  $V$  be a finite-dimensional vector space over  $\mathbb{R}$ , and let  $T: V \rightarrow V$  be a self-adjoint linear operator. Then  $T$  is positive semidefinite iff all of the eigenvalues of  $T$  are nonnegative.*

**Proof.** Suppose  $T$  is positive semidefinite. Let  $\lambda \in \mathbb{R}$  be an eigenvalue of  $T$ , and let  $x \in V$  be an eigenvector of  $T$  associated with  $\lambda$ . By definition we know that  $x \neq \vec{0}$ , so by the definition of positive semidefinite operators and properties of innerproducts we have

$$0 \leq \langle T(x), x \rangle = \langle \lambda x, x \rangle = \lambda \langle x, x \rangle.$$

By the definition of inner products we know that  $\langle x, x \rangle > 0$ . This implies that  $\lambda \geq 0$ .

Suppose all eigenvalues of  $T$  are nonnegative. By [FIS97, Theorem 6.17] there is an orthonormal basis  $\{x_1, \dots, x_n\}$  for  $V$  consisting of eigenvectors of  $T$ . For each  $i \in \{1, \dots, n\}$  let  $\lambda_i \in \mathbb{R}$  be the eigenvalue of  $T$  associated with  $x_i$ . Let  $v \in V$  be a nonzero vector. Then there are  $a_1, \dots, a_n \in \mathbb{R}$  such that  $v = \sum_{i=1}^n a_i x_i$ , so

$$\begin{aligned} \langle T(v), v \rangle &= \left\langle T \left( \sum_{i=1}^n a_i x_i \right), \sum_{j=1}^n a_j x_j \right\rangle = \left\langle \sum_{i=1}^n a_i T(x_i), \sum_{j=1}^n a_j x_j \right\rangle \\ &= \left\langle \sum_{i=1}^n a_i \lambda_i x_i, \sum_{j=1}^n a_j x_j \right\rangle = \sum_{i=1}^n \lambda_i a_i \sum_{j=1}^n a_j \langle x_i, x_j \rangle. \end{aligned}$$

Since  $\{x_1, \dots, x_n\}$  is an orthonormal basis, we know that  $\langle x_i, x_j \rangle$  is 0 if  $i \neq j$  and 1 if  $i = j$ , for all  $i, j \in \{1, \dots, n\}$ . Therefore

$$\sum_{i=1}^n \lambda_i a_i \sum_{j=1}^n a_j \langle x_i, x_j \rangle = \sum_{i=1}^n \lambda_i a_i^2.$$

For all  $i \in \{1, \dots, n\}$ , we know that  $a_i^2 \geq 0$ , and by hypothesis that  $\lambda_i \geq 0$ . Hence  $\langle T(v), v \rangle = \sum_{i=1}^n \lambda_i a_i^2 \geq 0$ . Since  $v \in V - \{\vec{0}\}$  was chosen arbitrarily, we have shown that  $T$  is positive semidefinite.  $\square$

**Lemma 2.2.8.** *Let  $V$  and  $W$  be finite-dimensional vector spaces over  $\mathbb{R}$ .*

- (1) *If  $U: V \rightarrow W$  is a linear transformation, then  $UU^*: W \rightarrow W$  is positive semidefinite.*
- (2) *If  $S, T: V \rightarrow V$  are positive semidefinite linear operators, then  $S + T: V \rightarrow V$  is positive semidefinite.*

**Proof.** (1) Let  $w \in W$  be a nonzero vector. First note that  $UU^*$  is self-adjoint. By the definition of adjoint operators and properties of the inner product, we have

$$\langle UU^*(w), w \rangle = \langle U^*(w), U^*(w) \rangle \geq 0.$$

Therefore  $UU^*$  is positive semidefinite.

(2) Let  $v \in V$  be a nonzero vector. Then by properties of inner products we have

$$\langle (S + T)(v), v \rangle = \langle S(v) + T(v), v \rangle = \langle S(v), v \rangle + \langle T(v), v \rangle \geq 0.$$

Therefore  $S + T$  is positive semidefinite.  $\square$

Finally, we present two results which may at first seem somewhat random, but which will be quite important later.

**Lemma 2.2.9.** *Let  $n$  be a positive integer, and let  $\mathbb{U}_n$  denote the  $n \times n$  matrix whose components are all 1. Then  $\text{Spec}(\mathbb{U}_n) = \{[0]^{n-1}, n\}$ .*

**Proof.** Let  $u$  denote the column vector of dimension  $n$  whose components are all 1, and for each  $i \in \{1, 2, \dots, n\}$  let  $b_i$  denote the column vector of dimension  $n$  whose  $i$ th component is 1 and whose other components are all 0. We will demonstrate that  $\beta = \{u, b_2, b_3, \dots, b_n\}$  is a basis for  $\mathbb{R}^n$ . We see easily that the set  $\{b_2, b_3, \dots, b_n\}$  is linearly independent, and it is also clear that no vector with a nonzero first coordinate, such as  $u$ , could possibly be in the span of this set. It follows by [FIS97, Theorem 1.8] that  $\beta$  is linearly independent. Since  $|\beta| = n = \dim(\mathbb{R}^n)$ , it must be that  $\beta$  is a basis for  $\mathbb{R}^n$ .

We will now rewrite the matrix  $\mathbb{U}_n$  relative to the basis  $\beta$ . Note that

$$\mathbb{U}_n u = \begin{pmatrix} n \\ n \\ \vdots \\ n \end{pmatrix} = nu \quad \text{and} \quad \mathbb{U}_n b_i = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = u$$

for all  $i \in \{2, 3, \dots, n\}$ . Therefore we have

$$[\mathbb{U}_n]_\beta = \begin{pmatrix} n & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

Since  $[\mathbb{U}_n]_\beta$  is an upper triangular matrix, we see that its eigenvalues are  $n$  and 0, with multiplicities 1 and  $n - 1$ , respectively, unless  $n = 1$ , in which case  $n$  is its only eigenvalue. Finally, since the eigenvalues of a linear transformation are invariant under a change of basis, these must be the eigenvalues, with their multiplicities, for  $\mathbb{U}_n$  as well.  $\square$

**Lemma 2.2.10.** *Let  $A_1, \dots, A_n$  be square matrices over a field  $F$ , and let*

$$M = \begin{pmatrix} A_1 & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & A_n \end{pmatrix}$$

*be a diagonal block matrix. Then*

$$\text{Spec}(M) = \text{Spec}(A_1) \cup_M \dots \cup_M \text{Spec}(A_n).$$

**Proof.** The eigenvalues of  $M$  are the roots of the characteristic polynomial  $\det(M - \lambda I)$  for  $M$ , where  $I$  is the identity matrix of the appropriate dimension. Note that

$$M - \lambda I = \begin{pmatrix} A_1 - \lambda I_1 & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & A_n - \lambda I_n \end{pmatrix},$$

where  $I_1, \dots, I_n$  are identity matrices of appropriate dimensions. By [FIS97, Exercise 20, page 218], we know that the determinant of a diagonal block matrix with two diagonal blocks is the product of the determinants of the two matrices that are the diagonal blocks. It follows inductively that the determinant of a diagonal block matrix is the product of

the determinants of all of the matrices that are the diagonal blocks. Hence  $\det(M - \lambda I) = \det(A_1 - \lambda I_1) \dots \det(A_n - \lambda I_n)$ , so therefore the roots of the characteristic polynomial of  $M$  is the multiset union of the roots of the characteristic polynomials of  $A_1, \dots, A_n$ . This proves the lemma.  $\square$

# 3

## Combinatorial Laplacians of Simplicial Complexes

### 3.1 Laplacians in Graph Theory

There are many equivalent definitions of a **graph** in the sense of graph theory. See [WES96, Section 1.1] for one such definition. We will denote the vertex set of a graph  $G$  by  $V(G)$  and the edge set by  $E(G)$ . As is standard, a **loop** is an edge that has a single vertex for both endpoints, a graph has **multiple edges** if two vertices in the graph have more than one edge between them, a **finite graph** is one with finite vertex and edge sets, and a **simple graph** is one with no loops or multiple edges. The **degree** of a vertex  $v$  in a graph  $G$ , denoted  $\deg_G(v)$ , is the number of edges in  $G$  containing  $v$ , and two vertices  $u$  and  $v$  are **adjacent** in  $G$ , denoted  $u \sim v$ , if there is an edge in  $G$  between  $u$  and  $v$ . The following definition comes from [CHU96, page 316].

**Definition.** Let  $G$  be a finite simple graph with  $V(G) = \{v_1, \dots, v_n\}$ . The **combinatorial Laplacian matrix** of  $G$ , denoted  $\mathcal{L}_G$ , is the  $n \times n$  matrix given by

$$(\mathcal{L}_G)_{ij} = \begin{cases} \deg_G(v_i), & \text{if } i = j \\ -1, & \text{if } v_i \text{ and } v_j \text{ are distinct and} \\ & v_i \sim v_j \\ 0, & \text{if } v_i \text{ and } v_j \text{ are distinct and} \\ & \text{non-adjacent} \end{cases}$$

for all  $i, j \in \{1, 2, \dots, n\}$ .

△

**Example 3.1.1.** Take the graph from Figure 3.1.1, the border of a triangle with an edge sticking off of one vertex. Vertices  $v_1$  and  $v_2$  have degree 2, vertex  $v_3$  has degree 3, and vertex  $v_4$  has degree 1. Vertices  $v_1$ ,  $v_2$ , and  $v_3$  are all adjacent to each other, and vertex

$v_3$  is adjacent to vertex  $v_4$ . Hence in this case we have

$$\mathcal{L}_G = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

◇

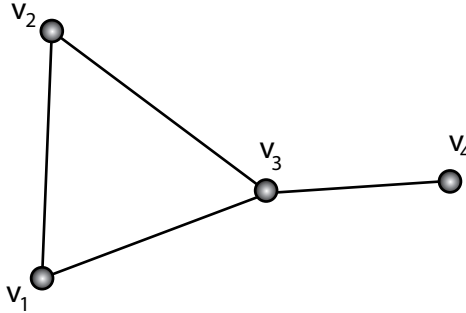


Figure 3.1.1.

The following standard lemma connects the combinatorial Laplacian matrix of a graph to the matrix representation of the boundary operator associated with that graph when the graph is oriented and seen as a 1-dimensional simplicial complex. (See [MUN84] for standard definitions of simplices, simplicial complexes, and associated ideas, including orientations, chains, and boundary operators of simplicial complexes.)

**Lemma 3.1.2.** *Let  $G$  be a finite simple graph, with  $V(G) = \{v_1, \dots, v_n\}$  and  $E(G) = \{e_1, e_2, \dots, e_m\}$ . Let the edges of  $G$  have arbitrary orientation, and let  $\mathcal{B}_1$  be a matrix representation of the boundary map  $\partial_1$  from the 1-chains over  $\mathbb{R}$  to the 0-chains over  $\mathbb{R}$  of the oriented graph  $G$ . Then*

$$\mathcal{L}_G = \mathcal{B}_1 \mathcal{B}_1^T.$$

This lemma is seen in Remark 3.3.5 to be a corollary of Theorem 3.3.4, a more general result.

## 3.2 Simplicial Complexes – Preliminaries

This section discusses simplicial complexes and introduces some short new definitions and results regarding them. Again, see [MUN84] for standard definitions of simplicial complexes and associated ideas, including orientations, chains, and boundary operators. Although [MUN84] uses chains over  $\mathbb{Z}$  for the most part, we will find it more convenient here to use chains over  $\mathbb{R}$ , as in [DURE]. For easy notation we will sometimes make use of the  $f$ -vector of a simplicial complex. If  $K$  is a finite simplicial complex of dimension  $d$ , then the  $f$ -vector of  $K$  is the  $(d+1)$ -dimensional vector  $(f_0(K), f_1(K), \dots, f_d(K))^T$ , where  $f_i(K)$  denotes the number of  $i$ -simplices in  $K$ , for each integer  $i$  with  $0 \leq i \leq d$ .



For our purposes, an oriented simplicial complex is one in which all simplices in the complex, except for vertices and  $\emptyset$ , are oriented. For any finite simplicial complex  $K$  and any nonnegative integer  $d$ , the collection of  $d$ -chains of  $K$ , denoted  $C_d$ , is a vector space over  $\mathbb{R}$ . (However, the chains still form a group, and we will follow tradition and refer to the set of chains of a given dimension as the chain group of that dimension.) A basis for  $C_d$  is given by the elementary chains associated with the  $d$ -simplices of  $K$ , so  $C_d$  has finite dimension  $f_d(K)$ . Also, if we look at elements of  $C_d$  as coordinates relative to this basis of elementary chains, we have the standard inner product on these coordinate vectors, and we see then that this basis of elementary chains is orthonormal. The  $d$ th boundary operator is a linear transformation  $C_d \rightarrow C_{d-1}$ , and it is denoted  $\partial_d$ . As is standard, for a simplicial complex of dimension  $k$  we define the  $(-1)$ -chain group and all chain groups of dimension greater than  $k$  to be the zero vector space.

**Definition.** Let  $K$  be a finite simplicial complex. Two distinct  $d$ -simplices  $\sigma_1, \sigma_2$  in  $K$  are **upper adjacent**, denoted  $\sigma_1 \sim_U \sigma_2$ , if both are faces of some  $(d+1)$ -simplex  $\tau$  in  $K$ , called their **common upper simplex**. The **upper degree** of a  $d$ -simplex  $\sigma$  in  $K$ , denoted  $\deg_U(\sigma)$ , is the number of  $(d+1)$ -simplices in  $K$  of which  $\sigma$  is a face.

Suppose  $K$  is oriented, and suppose  $\sigma_1$  and  $\sigma_2$  are  $d$ -simplices in  $K$  that are upper adjacent with common upper  $(d+1)$ -simplex  $\tau$ . We look at the signs of the coefficients of these two simplices in the sum  $\partial_{d+1}(\tau)$ . If the signs are the same, we say that  $\sigma_1$  and  $\sigma_2$  are **similarly oriented** with respect to  $\tau$ ; if the signs of the coefficients are different, we say the simplices are **dissimilarly oriented**.  $\triangle$

**Remark 3.2.1.** Note that the equality or inequality of the signs of the coefficients of two upper adjacent simplices in the sum representing the boundary of their common upper simplex does not depend on the orientation of the common upper simplex, but only on the orientations of the two upper adjacent simplices. Hence the similarity or dissimilarity of two upper adjacent simplices does not depend on the orientation of their common upper simplex. Also, it is standard to define a 0-simplex, typically called a vertex, as having only one choice of orientation. Hence, if two vertices in a simplicial complex are upper adjacent, then they are dissimilarly oriented with respect to their common upper 1-simplex.  $\diamond$

**Example.** For an example of the upper degree of a simplex, observe that in Figure 3.2.1, the upper degree of edge  $a$  is 3. As an example of upper adjacency, note that two edges in a simplicial complex are upper adjacent if they are both parts of a triangle in that complex. Intuitively, if the two edges are oriented so that they are “pointing” in the same direction around the triangle, then they are similarly oriented with respect to the triangle. If the edges are “pointing” in opposite directions around the triangle, then they are dissimilarly oriented. In Figure 3.2.2, edges  $a$  and  $b$  are upper adjacent and similarly oriented, while  $c$  and  $d$  are upper adjacent and dissimilarly oriented.  $\diamond$

**Remark.** Typically, the degree of a vertex in a graph is the number of edges in the graph containing it. In simple graphs, this definition of degree is seen to be a special case of the more general notion of upper degree given above, because a simple graph, once geometrically fixed, is really a 1-dimensional simplicial complex.  $\diamond$

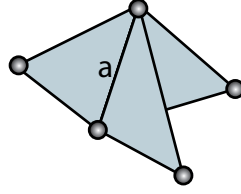


Figure 3.2.1.

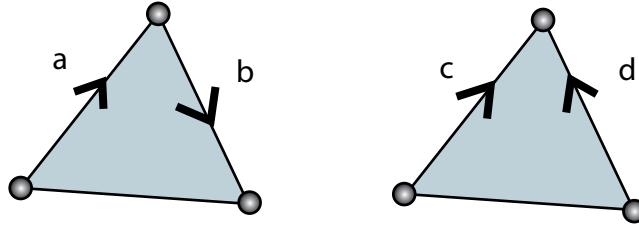


Figure 3.2.2.

**Lemma 3.2.2 (Uniqueness of Common Upper Simplex).** *Let  $K$  be a finite simplicial complex, and let  $\sigma_1, \sigma_2$  be two distinct  $d$ -simplices in  $K$ . If  $\sigma_1$  and  $\sigma_2$  are upper adjacent, then their common upper  $(d + 1)$ -simplex is unique.*

**Proof.** Suppose  $\tau_1$  and  $\tau_2$  are  $(d + 1)$ -simplices in  $K$  both of which contain both  $\sigma_1$  and  $\sigma_2$  as faces. Then  $\sigma_1 \cup \sigma_2 \subseteq \tau_1 \cap \tau_2$ . The definition of simplicial complexes requires that  $\tau_1 \cap \tau_2$  be a face of both  $\tau_1$  and  $\tau_2$ . Since  $\tau_1$  is a  $(d + 1)$ -simplex and  $\sigma_1$  and  $\sigma_2$  are distinct  $d$ -simplices, we see that the only face of  $\tau_1$  containing both  $\sigma_1$  and  $\sigma_2$  is  $\tau_1$  itself. Thus  $\tau_1 = \tau_1 \cap \tau_2$ , which implies that  $\tau_1$  is a face of  $\tau_2$ . Since  $\tau_1$  and  $\tau_2$  are both  $(d + 1)$ -simplices, this means that  $\tau_1 = \tau_2$ .  $\square$

**Definition.** Let  $K$  be a finite simplicial complex. Two distinct  $d$ -simplices  $\sigma_1, \sigma_2$  are **lower adjacent** in  $K$ , denoted  $\sigma_1 \sim_L \sigma_2$ , if both contain some nonempty  $(d - 1)$ -simplex  $\eta$  in  $K$  as a face. This  $(d - 1)$ -simplex  $\eta$  is called their **common lower simplex**. The **lower degree** of a  $d$ -simplex  $\sigma$  in  $K$ , denoted  $\deg_L(\sigma)$ , is the number of nonempty  $(d - 1)$ -simplices in  $K$  that are faces of  $\sigma$ .

Suppose  $K$  is oriented, and suppose that  $\sigma_1$  and  $\sigma_2$  are  $d$ -simplices in  $K$  that are lower adjacent with common lower  $(d - 1)$ -simplex  $\eta$ . We look at the signs of the coefficients of  $\eta$  in the sums  $\partial_d(\sigma_1)$  and  $\partial_d(\sigma_2)$ . If the signs are the same, we say that  $\eta$  is a **similar common lower simplex** of  $\sigma_1$  and  $\sigma_2$ ; if the signs of the coefficients are different, we say  $\eta$  is a **dissimilar common lower simplex**.  $\triangle$

**Remark 3.2.3.** We see immediately that if  $d > 0$ , then the lower degree of any  $d$ -simplex in any simplicial complex is  $d + 1$ . The lower degree of a vertex is 0. Since the only face that two vertices can have in common is the empty set, no two vertices can be lower adjacent.

Note that whether the signs of the coefficients of a common lower simplex in the sums representing the boundaries of two lower adjacent simplices are the same or different does not depend on the orientation of the common lower simplex, but only on the orientations

of the two lower adjacent simplices. Hence the similarity or dissimilarity of a common lower simplex does not depend on its orientation.  $\diamond$

**Example.** Two edges in a simplicial complex are lower adjacent if they share a vertex in that complex. Intuitively, if the two edges are oriented so that they are “pointing” either both towards the shared vertex or both away from the shared vertex, then the vertex is a similar common lower simplex. If the two edges are oriented so that one is “pointing” away from the shared vertex and the other is “pointing” towards it, then the vertex is a dissimilar common lower simplex. In Figure 3.2.2, edges  $a$  and  $b$  are lower adjacent with dissimilar common lower simplex, while  $c$  and  $d$  are lower adjacent with similar common lower simplex.  $\diamond$

**Lemma 3.2.4 (Uniqueness of Common Lower Simplex).** *Let  $\sigma_1$  and  $\sigma_2$  be distinct  $d$ -simplices of a simplicial complex  $K$ . If these two simplices are lower adjacent, then their common lower simplex is the intersection of the two simplices. Consequently, the common lower simplex of two simplices, if it exists, is unique.*

**Proof.** Suppose  $\eta$  is a common lower simplex of  $\sigma_1$  and  $\sigma_2$ . (Note that it follows from the definition of lower adjacency that  $\eta \neq \emptyset$  and  $d > 0$ .) Then  $\eta \subseteq \sigma_1 \cap \sigma_2$ . On the other hand, by the definition of a simplicial complex, we know that  $\sigma_1 \cap \sigma_2$  is a face of  $\sigma_1$ , and we also know that this must be a face of  $\sigma_1$  containing  $\eta$ . Note that since  $\eta$  is a  $(d-1)$ -simplex, there are precisely two faces of  $\sigma_1$  that contain  $\eta$ , namely  $\eta$  and  $\sigma_1$  itself. However, if  $\sigma_1 \cap \sigma_2 = \sigma_1$ , then  $\sigma_1$  is a face of  $\sigma_2$ , so  $\sigma_1 = \sigma_2$ , contradicting the fact that  $\sigma_1$  and  $\sigma_2$  are distinct. It follows that  $\eta = \sigma_1 \cap \sigma_2$ . Since any common lower simplex of two simplices is actually equal to the intersection of the simplices, we see that any two common lower simplices of two simplices must actually be identical.  $\square$

**Corollary 3.2.5.** *Two distinct  $d$ -simplices  $\sigma_1$  and  $\sigma_2$  of a finite simplicial complex  $K$  are lower adjacent iff  $\sigma_1 \cap \sigma_2$  is a nonempty  $(d-1)$ -simplex of  $K$ .*

**Lemma 3.2.6.** *Let  $K$  be a finite oriented simplicial complex, and let  $d$  be an integer with  $0 < d \leq \dim(K)$ . Let  $\sigma_1$  and  $\sigma_2$  be distinct and upper adjacent  $d$ -simplices of  $K$ , and let  $\tau$  be their common upper simplex. Then  $\sigma_1$  and  $\sigma_2$  are similarly oriented with respect to  $\tau$  iff they have a dissimilar common lower simplex.*

**Proof.** Without loss of generality, let  $\tau = \langle v_0, v_1, \dots, v_{d+1} \rangle$ , and let  $\sigma_1 = c_1 \langle v_1, v_2, \dots, v_{d+1} \rangle$  and  $\sigma_2 = c_2 \langle v_0, v_2, \dots, v_{d+1} \rangle$ , where  $c_1$  and  $c_2$  can each be either 1 or  $-1$ , depending on the orientations of  $\sigma_1$  and  $\sigma_2$ . Since  $d > 0$  we know that  $d+1 \geq 2$ , so we see that  $\eta = k \langle v_2, v_3, \dots, v_{d+1} \rangle$  is a common lower simplex of  $\sigma_1$  and  $\sigma_2$ , where  $k$  can be either 1 or  $-1$  depending on the orientation of  $\eta$ . Note that since  $c_1, c_2, k$  can each be 1 or  $-1$ , we know that  $\langle v_1, v_2, \dots, v_{d+1} \rangle = c_1 \sigma_1$ , and  $\langle v_0, v_2, \dots, v_{d+1} \rangle = c_2 \sigma_2$ , and  $\langle v_2, v_3, \dots, v_{d+1} \rangle = k \eta$ . Then

$$\begin{aligned} \partial_{d+1}(\tau) &= \partial_{d+1}(\langle v_0, v_1, \dots, v_{d+1} \rangle) \\ &= \langle v_1, v_2, \dots, v_{d+1} \rangle - \langle v_0, v_2, \dots, v_{d+1} \rangle + T = c_1 \sigma_1 - c_2 \sigma_2 + T \end{aligned}$$

where  $T$  represents the part of the sum with which we are not concerned now. Similarly, we see that

$$\partial_d(\sigma_1) = \partial_d(c_1 \langle v_1, v_2, \dots, v_{d+1} \rangle) = c_1 \langle v_2, v_3, \dots, v_{d+1} \rangle + T_1 = c_1 k \eta + T_1$$

and

$$\partial_d(\sigma_2) = \partial_d(c_2\langle v_0, v_2, \dots, v_{d+1} \rangle) = c_2\langle v_2, v_3, \dots, v_{d+1} \rangle + T_2 = c_2k\eta + T_2$$

where  $T_1$  and  $T_2$  represent the unimportant parts of the two sums.

( $\Rightarrow$ ) Suppose  $\sigma_1$  and  $\sigma_2$  are similarly oriented with respect to  $\tau$ . Then the signs of the coefficients of these two simplices in the sum  $\partial_{d+1}(\tau)$  must be the same, so it must be that  $c_1 = -c_2$ , so  $c_1$  and  $c_2$  have opposite signs. Then the coefficients of  $\eta$  in  $\partial_d(\sigma_1)$  and  $\partial_d(\sigma_2)$ , which are  $c_1k$  and  $c_2k$ , respectively, must have different signs. By definition, this means that  $\eta$  is a dissimilar common lower simplex of  $\sigma_1$  and  $\sigma_2$ .

( $\Leftarrow$ ) Suppose  $\sigma_1$  and  $\sigma_2$  are dissimilarly oriented with respect to  $\tau$ . Then it must be that  $c_1$  and  $c_2$  have the same sign, so then  $c_1k$  and  $c_2k$  both have the same sign, which means that  $\eta$  is a similar common lower simplex of  $\sigma_1$  and  $\sigma_2$ . The contrapositive of this statement is, if  $\eta$  is a similar common lower simplex of  $\sigma_1$  and  $\sigma_2$  then  $\sigma_1$  and  $\sigma_2$  are similarly oriented with respect to  $\tau$ .  $\square$

In the course of the above proof, we proved the following intuitive corollary.

**Corollary 3.2.7.** *Let  $d > 0$  be an integer. If two distinct  $d$ -simplices of a finite simplicial complex are upper adjacent, then they are also lower adjacent.*

### 3.3 Laplacians of Simplicial Complexes

Note that the matrices of the sole linear operators between trivial vector spaces, from the trivial vector space to a vector space of dimension  $n$ , and from a vector space of dimension  $n$  to the trivial vector space are the  $1 \times 1$  zero matrix, a column vector with  $n$  entries all of which are 0, and a row vector with  $n$  entries all of which are 0, respectively. (These are the matrices of the boundary operator  $\partial_d$  of a simplicial complex  $K$  in the cases where  $d > \dim(K) + 1$ , where  $d = \dim(K) + 1$ , and where  $d = 0$ , respectively.)

For each boundary operator  $\partial_d: C_d \rightarrow C_{d-1}$  of  $K$ , we let  $\mathcal{B}_d$  be the matrix representation of this operator relative to the standard bases for  $C_d$  and  $C_{d-1}$  with some orderings given to them. We see that the number of rows in  $\mathcal{B}_d$  is the number of  $(d-1)$ -simplices in  $K$ , and the number of columns is the number of  $d$ -simplices. Associated with the boundary operator  $\partial_d$  is its adjoint operator  $\partial_d^*: C_{d-1} \rightarrow C_d$ . From [FIS97, Theorem 6.10], we know that the transpose of the matrix for the  $d$ th boundary operator relative to the standard orthonormal basis of elementary chains with some ordering,  $\mathcal{B}_d^T$ , is the matrix representation of the  $d$ th adjoint boundary operator,  $\partial_d^*$  with respect to this same ordered basis.

It is worth noting that the  $d$ th adjoint boundary operator of a finite oriented simplicial complex  $K$  is in fact the same as the  $d$ th coboundary operator  $\delta_d: C^{d-1}(K; \mathbb{R}) \rightarrow C^d(K; \mathbb{R})$  given in [FIS97, page 6], under the isomorphism  $C^d(K; \mathbb{R}) = \text{Hom}(C_d(K), \mathbb{R}) \cong C_d(K)$ .

The composition of two composable linear maps is a linear map. Two linear maps with identical domains and codomains can be added by adding their values on any element in their domain, and the sum of two such maps is clearly another linear map. The following definition comes from [DURE].

**Definition.** Let  $K$  be a finite oriented simplicial complex, and let  $d \geq 0$  be an integer. The  $d$ th combinatorial Laplacian is the linear operator  $\Delta_d: C_d \rightarrow C_d$  given by

$$\Delta_d = \partial_{d+1} \circ \partial_{d+1}^* + \partial_d^* \circ \partial_d.$$

For convenience, we use the notations  $\Delta_d^{UP} = \partial_{d+1} \circ \partial_{d+1}^*$  and  $\Delta_d^{DN} = \partial_d^* \circ \partial_d$ , so that  $\Delta_d = \Delta_d^{UP} + \Delta_d^{DN}$ .

△

The  $d$ th Laplacian matrix of  $K$ , denoted  $\mathcal{L}_d$ , relative to some orderings of the standard bases for  $C_d$  and  $C_{d-1}$  of  $K$ , is the matrix representation of  $\Delta_d$ . Observe that

$$\mathcal{L}_d = \mathcal{B}_{d+1} \mathcal{B}_{d+1}^T + \mathcal{B}_d^T \mathcal{B}_d.$$

As above, for convenience, we use the notations  $\mathcal{L}_d^{UP} = \mathcal{B}_{d+1} \mathcal{B}_{d+1}^T$  and  $\mathcal{L}_d^{DN} = \mathcal{B}_d^T \mathcal{B}_d$ , so that  $\mathcal{L}_d = \mathcal{L}_d^{UP} + \mathcal{L}_d^{DN}$ .

**Example 3.3.1.** Let  $K$  be the oriented simplicial complex given in Figure 3.3.1. Then we calculate that

$$\mathcal{B}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \text{ and } \mathcal{B}_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore

$$\begin{aligned} \mathcal{L}_1 &= \mathcal{B}_2 \mathcal{B}_2^T + \mathcal{B}_1^T \mathcal{B}_1 \\ &= \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & -1 \\ 1 & 0 & -1 & 2 \end{pmatrix}. \end{aligned}$$

◇

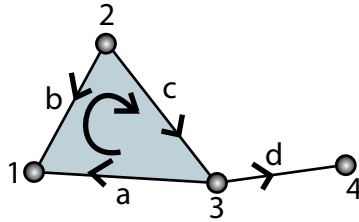


Figure 3.3.1.

Note that  $\partial_0$  is the zero map for any simplicial complex, so  $\mathcal{B}_0$  is a zero matrix. We see then that  $\partial_0^* \circ \partial_0$  is the zero map, and  $\mathcal{L}_0^{DN} = \mathcal{B}_0^T \mathcal{B}_0$  is a zero matrix, so  $\Delta_0 = \partial_1 \circ \partial_1^*$  and  $\mathcal{L}_0 = \mathcal{L}_d^{UP}$ . Referring back to Lemma 3.1.2, yet to be demonstrated, we see that our Laplacian matrix for simplicial complexes is a generalization of the combinatorial Laplacian matrix defined in graph theory, because finite simple graphs can be seen as simplicial complexes of dimension 1, embedded in some Euclidean space.

We now present several results that greatly ease the computation of the Laplacian matrix of a simplicial complex.

**Proposition 3.3.2.** *Let  $K$  be a finite oriented simplicial complex, and let  $d$  be an integer with  $0 \leq d \leq \dim(K)$ . Let  $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$  be the  $d$ -simplices of  $K$ , and let  $\{\tau_1, \tau_2, \dots, \tau_m\}$  be the  $(d+1)$ -simplices of  $K$ . Let  $i, j \in \{1, 2, \dots, n\}$ . Then*

$$(\mathcal{L}_d^{UP})_{ij} = \begin{cases} \deg_U(\sigma_i), & \text{if } i = j \\ 1, & \text{if } i \neq j \text{ and } \sigma_i \text{ and } \sigma_j \text{ are upper adjacent} \\ & \text{and oriented similarly} \\ -1, & \text{if } i \neq j \text{ and } \sigma_i \text{ and } \sigma_j \text{ are upper adjacent} \\ & \text{and oriented dissimilarly} \\ 0, & \text{if } i \neq j \text{ and } \sigma_i \text{ and } \sigma_j \text{ are not upper adjacent.} \end{cases}$$

**Proof.** First, if  $d = \dim(K)$ , then  $\partial_{d+1}: C_{d+1} \rightarrow C_d$  is the zero map from the trivial vector space to another vector space, so we see that  $\mathcal{L}_{d+1}^{UP}$  must be a zero matrix. Since in this case there are no  $(d+1)$ -simplices in  $K$ , no  $d$ -simplices are upper adjacent in  $K$ , so the proposition follows.

Now, suppose  $d < \dim(K)$ . The  $ij$ th component of  $\mathcal{L}_{d+1}^{UP} = \mathcal{B}_{d+1} \mathcal{B}_{d+1}^*$  is the standard dot product of the  $i$ th and  $j$ th rows of  $\mathcal{B}_{d+1}$ . Let  $X$  and  $Y$  denote these rows, respectively. As before, we will refer to the individual products of the components of  $X$  and  $Y$  in their dot product as summands.

Suppose  $i = j$ . Let  $\tau_k$  be a  $(d+1)$ -simplex in  $K$ . If  $\sigma_i$  is a face of  $\tau_k$ , then the  $k$ th component of  $X$  is either 1 or  $-1$ , depending on the orientation of  $\sigma_i$ , so the  $k$ th summand of  $X \cdot X$  is 1. If  $\sigma_i$  is not a face of  $\tau_k$ , then the  $k$ th component of  $X$  is 0, so the  $k$ th summand of  $X \cdot X$  is 0. It follows then that  $X \cdot X$  is the same as the number of  $(d+1)$ -simplices in  $K$  of which  $\sigma_i$  is a face, which is of course the upper degree of  $\sigma_i$  in  $K$ .

Suppose  $i \neq j$ . Let  $\tau_k$  be a  $(d+1)$ -simplex in  $K$ . Suppose  $\sigma_i$  and  $\sigma_j$  are both faces of  $\tau_k$ . If  $\sigma_i$  and  $\sigma_j$  are similarly oriented, then the  $k$ th components of  $X$  and  $Y$  are either both 1 or both  $-1$ , so either way the  $k$ th summand of  $X \cdot Y$  is 1. If  $\sigma_i$  and  $\sigma_j$  are dissimilarly oriented, then of the  $k$ th components of  $X$  and  $Y$ , one is 1 and the other is  $-1$ , so the  $k$ th summand in  $X \cdot Y$  is  $-1$ . If  $\sigma_i$  is not a face of  $\tau_k$ , then the  $k$ th component of  $X$  is 0. Similarly for  $\sigma_j$ , so if either  $\sigma_i$  or  $\sigma_j$  is not a face of  $\tau_k$ , then the  $k$ th summand of  $X \cdot Y$  is 0.

We know by Lemma 3.2.2 that there is at most one  $(d+1)$ -simplex in  $K$  containing both  $\sigma_i$  and  $\sigma_j$  as faces. Therefore, if  $\sigma_i$  and  $\sigma_j$  are upper adjacent, then a single summand of  $X \cdot Y$  is either 1 or  $-1$  and all the other summands are 0, so  $(\mathcal{L}_{d+1}^{UP})_{ij} = X \cdot Y$  is either 1 or  $-1$ , depending on whether the two simplices are oriented similarly or dissimilarly, respectively. If  $\sigma_i$  and  $\sigma_j$  are not upper adjacent, then all the summands of  $X \cdot Y$  are 0, so  $(\mathcal{L}_{d+1}^{UP})_{ij} = X \cdot Y = 0$ .  $\square$

**Proposition 3.3.3.** *Let  $K$  be a finite oriented simplicial complex, and let  $d$  be an integer with  $0 \leq d \leq \dim(K)$ . Let  $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$  be the  $d$ -simplices of  $K$ . Let  $i, j \in \{1, 2, \dots, n\}$ . Then*

$$(\mathcal{L}_d^{DN})_{ij} = \begin{cases} \deg_L(\sigma_i), & \text{if } i = j \\ 1, & \text{if } i \neq j \text{ and } \sigma_i \text{ and } \sigma_j \text{ have a similar} \\ & \text{common lower simplex} \\ -1, & \text{if } i \neq j \text{ and } \sigma_i \text{ and } \sigma_j \text{ have a dissimilar} \\ & \text{common lower simplex} \\ 0, & \text{if } i \neq j \text{ and } \sigma_i \text{ and } \sigma_j \text{ are not lower adjacent.} \end{cases}$$

**Proof.** First, if  $d = 0$ , then since  $\partial_0: C_0 \rightarrow C_{-1}$  is the zero map from a finite dimensional vector space to the trivial vector space, we know that  $\mathcal{L}_d^{DN}$  must be a zero matrix. Since no vertices of a simplicial complex are lower adjacent, the proposition follows.

Suppose  $d > 0$ . The  $ij$ th component of  $\mathcal{L}_d^{DN} = \mathcal{B}_d^* \mathcal{B}_d$  is the standard dot product of the  $i$ th and  $j$ th columns of  $\mathcal{B}_d$ . Let  $X$  and  $Y$  denote these columns, respectively, and let  $\{\eta_1, \eta_2, \dots, \eta_m\}$  be the  $(d-1)$ -simplices of  $K$ .

Suppose  $i = j$ . Let  $\eta_k$  be a  $(d-1)$ -simplex of  $K$ . If  $\sigma_i$  contains  $\eta_k$  as a face, then the  $k$ th component of  $X$  is either 1 or  $-1$ , depending on the orientation of  $\eta_k$ , so the  $k$ th summand of  $X \cdot X$  is 1. If  $\sigma_i$  does not contain  $\eta_k$  as a face, then the  $k$ th component of  $X$  is 0, so the  $k$ th summand of  $X \cdot X$  is 0. It follows then that  $(\mathcal{L}_d^{DN})_{ij} = X \cdot X$  is the same as the number of  $(d-1)$ -faces of  $\sigma_i$ , which is the lower degree of  $\sigma_i$  in  $K$ .

Suppose  $i \neq j$ . Let  $\eta_k$  be a  $(d-1)$ -simplex of  $K$ . Suppose  $\sigma_i \cap \sigma_j = \eta_k$ , meaning by Corollary 3.2.5 that  $\sigma_i$  and  $\sigma_j$  are lower adjacent. If  $\eta_k$  is a similar common lower simplex, then the  $k$ th components of  $X$  and  $Y$  are either both 1 or both  $-1$ , so either way the  $k$ th summand of  $X \cdot Y$  is 1. If  $\eta_k$  is a dissimilar common lower simplex, then of the  $k$ th components of  $X$  and  $Y$ , one is 1 and the other is  $-1$ , so the  $k$ th summand in  $X \cdot Y$  is  $-1$ . If  $\eta_k$  is not a face of  $\sigma_i$ , then the  $k$ th component of  $X$  is 0. Similarly for  $\sigma_j$ , so if  $\eta_k$  is not a common lower simplex of  $\sigma_i$  and  $\sigma_j$ , then the  $k$ th summand of  $X \cdot Y$  is 0.

We know by Lemma 3.2.4 that there is at most one  $(d-1)$ -simplex in  $K$  that is a face of both  $\sigma_i$  and  $\sigma_j$ . Therefore, if  $\sigma_i$  and  $\sigma_j$  have a similar common lower simplex, then a single summand of  $X \cdot Y$  is 1 and all the other summands are 0, so  $(\mathcal{L}_d^{DN})_{ij} = X \cdot Y = 1$ . If  $\sigma_i$  and  $\sigma_j$  have a dissimilar common lower simplex, then a single summand of  $X \cdot Y$  is  $-1$  and all the other summands are 0, so  $(\mathcal{L}_d^{DN})_{ij} = X \cdot Y = -1$ . If  $\sigma_i$  and  $\sigma_j$  are not lower adjacent, then all the summands of  $X \cdot Y$  are 0, so  $(\mathcal{L}_d^{DN})_{ij} = X \cdot Y = 0$ .  $\square$

**Theorem 3.3.4 (Laplacian Matrix Theorem).** *Let  $K$  be a finite oriented simplicial complex, let  $d$  be an integer with  $0 \leq d \leq \dim(K)$ , and let  $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$  denote the  $d$ -simplices of  $K$ . Let  $i, j \in \{1, 2, \dots, n\}$ .*

(1) *If  $d = 0$ , then*

$$(\mathcal{L}_d)_{ij} = \begin{cases} \deg_U(\sigma_i), & \text{if } i = j \\ -1, & \text{if } \sigma_i \text{ and } \sigma_j \text{ are distinct and} \\ & \text{upper adjacent} \\ 0, & \text{if } \sigma_i \text{ and } \sigma_j \text{ are distinct and} \\ & \text{not upper adjacent.} \end{cases}$$

(2) If  $d > 0$ , then

$$(\mathcal{L}_d)_{ij} = \begin{cases} \deg_U(\sigma_i) + d + 1, & \text{if } i = j \\ 1, & \text{if } i \neq j \text{ and } \sigma_i \text{ and } \sigma_j \text{ are not upper adjacent} \\ & \text{but have a similar common lower simplex} \\ -1, & \text{if } i \neq j \text{ and } \sigma_i \text{ and } \sigma_j \text{ are not upper adjacent} \\ & \text{but have a dissimilar common lower simplex} \\ 0, & \text{if } i \neq j \text{ and either } \sigma_i \text{ and } \sigma_j \text{ are upper adjacent} \\ & \text{or are not lower adjacent.} \end{cases}$$

**Proof.** (1) Suppose  $d = 0$ . We remarked at the beginning of this section that  $\mathcal{L}_0 = \mathcal{B}_1 \mathcal{B}_1^T$ . In Remark 3.2.1 we noted that since vertices have only one choice of orientation, any two vertices that are upper adjacent are dissimilarly oriented. We see that part (1) of this theorem follows directly from Proposition 3.3.2.

(2) Suppose  $d > 0$ . If  $i = j$ , then by Proposition 3.3.2 and Proposition 3.3.3 we know that  $(\mathcal{L}_d)_{ii} = (\mathcal{L}_d^{UP})_{ii} + (\mathcal{L}_d^{DN})_{ii} = \deg_U(\sigma_i) + \deg_L(\sigma_i)$ . Since every simplex of dimension  $d > 0$  has exactly  $d + 1$   $(d - 1)$ -faces, we see that  $(\mathcal{L}_d)_{ii} = \deg_U(\sigma_i) + d + 1$ .

Suppose  $i \neq j$ . If  $\sigma_i$  and  $\sigma_j$  are not upper adjacent but have a similar common lower simplex, then by Proposition 3.3.2 and Proposition 3.3.3 we know that  $(\mathcal{L}_d)_{ij} = (\mathcal{L}_d^{UP})_{ij} + (\mathcal{L}_d^{DN})_{ij} = 0 + 1 = 1$ . If  $\sigma_i$  and  $\sigma_j$  are not upper adjacent but have a dissimilar common lower simplex, then by these same propositions we know that  $(\mathcal{L}_d)_{ij} = (\mathcal{L}_d^{UP})_{ij} + (\mathcal{L}_d^{DN})_{ij} = 0 + (-1) = -1$ .

Suppose  $\sigma_i$  and  $\sigma_j$  are upper adjacent. If they are similarly oriented, then we know by Lemma 3.2.6 that they have a dissimilar common lower simplex, so by the same two propositions used above, we have that  $(\mathcal{L}_d)_{ij} = (\mathcal{L}_d^{UP})_{ij} + (\mathcal{L}_d^{DN})_{ij} = 1 + (-1) = 0$ . On the other hand, if they are dissimilarly oriented, then they have a similar common lower simplex, so  $(\mathcal{L}_d)_{ij} = (\mathcal{L}_d^{UP})_{ij} + (\mathcal{L}_d^{DN})_{ij} = (-1) + 1 = 0$ .

Finally, if  $\sigma_j$  and  $\sigma_i$  are not lower adjacent, then we know by the contrapositive of Corollary 3.2.7 that they are not upper adjacent, so then by the same propositions referred to above, we have that  $(\mathcal{L}_d)_{ij} = (\mathcal{L}_d^{UP})_{ij} + (\mathcal{L}_d^{DN})_{ij} = 0 + 0 = 0$ .  $\square$

The Laplacian Matrix Theorem confirms our calculation of Example 3.3.1. (Or, our calculation of Example 3.3.1 confirms the Laplacian Matrix Theorem, depending on your point of view.)

**Remark 3.3.5.** Since a finite simple graph  $G$  can be viewed as a simplicial complex of dimension 1, we conclude from Theorem 3.3.4 that the matrix  $\mathcal{L}_G$  from the definition in Section 3.1 is the same as the zero Laplacian matrix of  $G$  as a simplicial complex. Since the definition of the Laplacian matrix of simplicial complexes is in terms of the boundary operator, we see that Lemma 3.1.2 follows from Theorem 3.3.4 as a corollary.  $\diamond$

**Remark 3.3.6.** Note that the above theorem implies that the value of the  $d$ th Laplacian matrix of a simplicial complex really depends at most on the orientations of the  $d$ -simplices of the complex, and not on orientations of simplices of other dimensions.  $\diamond$

From the Theorem 3.3.4, we deduce the following Corollary. Note that part (1) of this Corollary is also a formula about the graph theory Laplacian matrix, since the graph



theory Laplacian is the same as the 0-dimensional Laplacian for simplicial complexes. The formula in part (1) is a well-known equation in the study of graph theory Laplacians. It came from [CHU96, page 317], and served as the inspiration for the formula of part (2).

**Corollary 3.3.7.** *Let  $K$  be a finite oriented simplicial complex.*

(1) *Let  $v_1, \dots, v_m$  be the vertices of  $K$ , and let  $i \in \{1, \dots, m\}$ . Then*

$$\Delta_0(v_i) = \sum_{v_j \sim_U v_i} (v_i - v_j).$$

(2) *Let  $d$  be an integer with  $0 < d \leq \dim(K)$ , let  $\sigma_1, \dots, \sigma_n$  be the oriented  $d$ -simplices of  $K$ , and let  $i \in \{1, \dots, n\}$ . Then*

$$\Delta_d(\sigma_i) = \sum_{\sigma_j \sim_L \sigma_i} (\sigma_i + s_{ij}\sigma_j) + \sum_{\sigma_k \sim_U \sigma_i} (\sigma_i - s_{ik}\sigma_k),$$

where  $s_{ij}$  is 1 if  $\sigma_i$  and  $\sigma_j$  have a similar common lower simplex, and  $-1$  if they have a dissimilar common lower simplex, for all  $i, j \in \{1, \dots, n\}$ .

**Proof.** (1) Theorem 3.3.4 tells us exactly what each entry of  $\mathcal{L}_0$  looks like, and the vertex  $v_i$  can be represented by the  $i$ th standard basis vector for  $\mathbb{R}^m$ , which we will denote  $e_i$ . Then  $\Delta_0(v_i)$  is the chain represented by the vector  $\mathcal{L}_0 e_i$ . This vector is the  $i$ th column of  $\mathcal{L}_0$ , so we see that

$$\Delta_0(v_i) = \deg_U(v_i)v_i - \sum_{v_j \sim_U v_i} v_j.$$

The number of  $j \in \{1, \dots, m\} - \{i\}$  for which  $v_i$  is upper adjacent to  $v_j$  is precisely the upper degree of  $v_i$ , so this formula reduces to

$$\Delta_0(v_i) = \sum_{v_j \sim_U v_i} (v_i - v_j).$$

(2) As in part (1), we see that  $\Delta_d(\sigma_i)$  is the chain represented by the vector  $\mathcal{L}_d e_i$ , where  $e_i \in \mathbb{R}^n$  is the  $i$ th standard basis vector. Again, this vector is the  $i$ th column of  $\mathcal{L}_d$ . From Theorem 3.3.4, we deduce that

$$\Delta_d(\sigma_i) = (\deg_U(\sigma_i) + d + 1)\sigma_i + \sum_{\sigma_j \sim_L \sigma_i} s_{ij}\sigma_j - \sum_{\sigma_k \sim_U \sigma_i} s_{ik}\sigma_k,$$

because since any two upper adjacent simplices are also lower adjacent, subtracting the two sums will cancel all terms containing a simplex  $\sigma_j$  that is upper adjacent to  $\sigma_i$ , which is precisely what is required since the  $j$ th entry of the  $i$ th column of  $\mathcal{L}_d$  is 0 if  $\sigma_j$  is upper adjacent to  $\sigma_i$ . The coefficients  $s_{ij}$  account for the signs of the remaining entries, depending on the similarity or dissimilarity of the relevant common lower simplices. Note that the number of  $\sigma_j$  that are lower adjacent to  $\sigma_i$  is precisely  $d + 1$ , and the number of  $\sigma_k$  that are upper adjacent to  $\sigma_i$  is precisely  $\deg_U(\sigma_i)$ . Therefore the formula we calculated reduces to the desired formula, namely

$$\Delta_d(\sigma_i) = \sum_{\sigma_j \sim_L \sigma_i} (\sigma_i + s_{ij}\sigma_j) + \sum_{\sigma_k \sim_U \sigma_i} (\sigma_i - s_{ik}\sigma_k).$$

□

**Remark 3.3.8.** Even though we are using chains over  $\mathbb{R}$ , the information presented and proved in Section 2.1 shows that all of our work, including defining the Laplacian operator and its matrix, could just as well be done over  $\mathbb{Z}$  instead. Over  $\mathbb{Z}$ , the chains of a simplicial complex form a free abelian group, and the boundary operator and Laplacian operators are both homomorphisms. As detailed in Section 2.1, free abelian groups are algebraic structures that are like enough to vector spaces to allow matrix representations of homomorphisms between them, and also to define adjoint homomorphisms whose matrix representations are the transposes of the matrices of the original functions.

One reason we use  $\mathbb{R}$  here is that it allows us to use results from linear algebra directly on the Laplacian operator, which over  $\mathbb{R}$  is a linear operator, rather than using these same results indirectly on a matrix representation of a homomorphism instead of on the homomorphism itself. Another very important reason to use vector spaces instead of free abelian groups is that even though we can define the eigenvalues of a homomorphism between free abelian groups to be the eigenvalues of its matrix, the meaning of eigenvalues and eigenvectors for the operator itself may be lost, because the vector representations of elements of a free abelian group have only integer entries. Any non-integer eigenvalues have little of their usual meaning for the homomorphism, and of course non-integer eigenvectors do not exist in the chain groups.  $\diamond$

### 3.4 Reduced Laplacians of Simplicial Complexes

Let  $K$  be a finite oriented simplicial complex. It is standard to take  $\emptyset$  to be a  $(-1)$ -simplex of  $K$ , and suppose that every simplex in  $K$  contains  $\emptyset$  as a face. Normally, the set of  $(-1)$ -chains in  $K$  is defined to be the trivial vector space. Suppose we define the set of  $(-1)$ -chains to be the vector space with singleton basis containing the elementary chain corresponding to  $\emptyset$ . Then this vector space is isomorphic to  $\mathbb{R}$ . In algebraic topology it is sometimes useful to regard the  $(-1)$ -chain group to be isomorphic to  $\mathbb{R}$ , and look at something called the reduced homology groups of  $K$ , as in [MUN84]. We will call the nontrivial vector space of chains with basis  $\{\emptyset\}$  the **augmented chain group of dimension  $(-1)$** , denoted  $\tilde{C}_{-1}$ .

In this context, for all integers  $d > 0$  we will speak of the **augmented chain group of dimension  $d$** , denoted  $\tilde{C}_d$ , defined to be identical to  $C_d$ , the usual chain group of dimension  $d$ . It is also standard to define an **augmented boundary operator of dimension 0**, denoted  $\tilde{\partial}_0: \tilde{C}_0 \rightarrow \tilde{C}_{-1}$ , as the linear operator given by  $\tilde{\partial}_0(v) = \emptyset$  for all  $v \in \tilde{C}_0 = C_0$ . Similar to augmented chain groups, for all integers  $d > 0$  we will speak of the **augmented boundary operator of dimension  $d$** , denoted  $\tilde{\partial}_d$ , defined to be identical to  $\partial_d$ , the usual boundary operator of dimension  $d$ . Similarly to our nonreduced definitions, the adjoint operator of  $\tilde{\partial}_d$  will be denoted  $\tilde{\partial}_d^*$ . The **reduced combinatorial Laplacian of dimension  $d$** , denoted  $\tilde{\Delta}_d$ , is the homomorphism from  $\tilde{C}_d$  to  $\tilde{C}_d$  given by

$$\tilde{\Delta}_d = \tilde{\partial}_{d+1} \circ \tilde{\partial}_{d+1}^* + \tilde{\partial}_d^* \circ \tilde{\partial}_d.$$

Similarly to the unreduced case, for convenience we use the notations  $\tilde{\Delta}_d^{UP} = \tilde{\partial}_{d+1} \tilde{\partial}_{d+1}^T$  and  $\tilde{\Delta}_d^{DN} = \tilde{\partial}_d^T \tilde{\partial}_d$ , so that  $\tilde{\Delta}_d = \tilde{\Delta}_d^{UP} + \tilde{\Delta}_d^{DN}$ .

For all integers  $d \geq 0$ , we let  $\tilde{\mathcal{B}}_d$  denote the standard matrix representation of  $\tilde{\partial}_d$ . As in the unreduced case, we know that  $\tilde{\mathcal{B}}_d^T$  is a matrix representation of  $\tilde{\partial}_d^*$ . The **reduced Laplacian matrix of dimension  $d$** , denoted  $\tilde{\mathcal{L}}_d$ , is the matrix representation of  $\tilde{\Delta}_d$ , and we see that

$$\tilde{\mathcal{L}}_d = \tilde{\mathcal{B}}_{d+1} \tilde{\mathcal{B}}_{d+1}^T + \tilde{\mathcal{B}}_d^T \tilde{\mathcal{B}}_d.$$

Similarly to the unreduced case, for convenience we use the notations  $\tilde{\mathcal{L}}_d^{UP} = \tilde{\mathcal{B}}_{d+1} \tilde{\mathcal{B}}_{d+1}^T$  and  $\tilde{\mathcal{L}}_d^{DN} = \tilde{\mathcal{B}}_d^T \tilde{\mathcal{B}}_d$ , so that  $\tilde{\mathcal{L}}_d = \tilde{\mathcal{L}}_d^{UP} + \tilde{\mathcal{L}}_d^{DN}$ .

**Remark 3.4.1.** Suppose that  $K$  contains  $n$  vertices. Then  $\tilde{\mathcal{B}}_0$  is a row vector with  $n$  entries all of which are 1. Hence  $\tilde{\mathcal{B}}_0^T \tilde{\mathcal{B}}_0$  is an  $n \times n$  matrix all of whose entries are 1, which we denoted  $\mathbb{U}_n$  in Lemma 2.2.9. Then

$$\tilde{\mathcal{L}}_0 = \mathcal{L}_0 + \mathbb{U}_n.$$

Also, we see immediately that for any finite oriented simplicial complex  $K$  it must be that  $\tilde{\Delta}_d = \Delta_d$ , and so also  $\tilde{\mathcal{L}}_d = \mathcal{L}_d$ , for all integers  $d > 0$ .  $\diamond$

We will now reformulate our definitions and results from Sections 3.2 and 3.3 in terms of reduced Laplacians.

**Definition.** Let  $K$  be a finite simplicial complex. Two distinct  $d$ -simplices  $\sigma_1, \sigma_2$  are **reduced lower adjacent** in  $K$  if both contain some (possibly empty)  $(d-1)$ -simplex  $\eta$  in  $K$  as a face. This  $(d-1)$ -simplex  $\eta$  is called their **reduced common lower simplex**. The **reduced lower degree** of a  $d$ -simplex  $\sigma$  in  $K$ , denoted  $\deg_{\tilde{\mathcal{L}}}(\sigma)$ , is the number of (possibly empty)  $(d-1)$ -simplices in  $K$  that are faces of  $\sigma$ .

Suppose  $K$  is oriented and that  $\sigma_1$  and  $\sigma_2$  are  $d$ -simplices in  $K$  that are lower adjacent with common lower  $(d-1)$ -simplex  $\eta$ . We look at the signs of the coefficients of  $\eta$  in the sums  $\partial_d(\sigma_1)$  and  $\partial_d(\sigma_2)$ . If the signs are the same, we say that  $\eta$  is a **similar reduced common lower simplex** of  $\sigma_1$  and  $\sigma_2$ ; if the signs of the coefficients are different, we say  $\eta$  is a **dissimilar reduced common lower simplex**.  $\triangle$

As before, if  $d > 0$  then the reduced lower degree of any  $d$ -simplex in any simplicial complex is  $d+1$ ; however, now any two vertices have a reduced common lower simplex, namely  $\emptyset$ . Hence, for all integers  $d \geq 0$ , the reduced lower degree of any  $d$ -simplex in a simplicial complex is  $d+1$ . Also, since we assume the empty set has only one orientation, we see that  $\emptyset$  is a similar reduced common lower simplex of any two vertices in a simplicial complex.

For any dimension greater than 0, the concepts of reduced lower adjacency and similar or dissimilar reduced common lower simplices mean exactly the same as their analogues in un-reduced Laplacians. Concepts of upper adjacency are unaffected when considering reduced Laplacians.

We will find that the same results that we proved in Section 3.3 in the study of un-reduced Laplacians hold true for the reduced case, except that we gain greater generality in that all of our results now hold for the 0-dimensional case as well, whereas previously some of them did not.

**Lemma 3.4.2 (Uniqueness of Reduced Common Lower Simplex).** *Let  $\sigma_1$  and  $\sigma_2$  be distinct  $d$ -simplices of a simplicial complex  $K$ . If these two simplices are reduced lower*

adjacent, then their reduced common lower simplex is the intersection of the two simplices. Consequently, the reduced common lower simplex of two simplices, if it exists, is unique.

**Proof.** For  $d > 0$ , the proof of this lemma is the same as for Lemma 3.2.4 in the un-reduced case. If  $d = 0$ , then we note that the intersection of any two vertices is the empty set, which is of course a reduced common lower simplex of the two vertices.  $\square$

**Corollary 3.4.3.** *Two distinct  $d$ -simplices  $\sigma_1$  and  $\sigma_2$  of a finite simplicial complex  $K$  are reduced lower adjacent iff  $\sigma_1 \cap \sigma_2$  is a  $(d - 1)$ -simplex of  $K$ .*

**Lemma 3.4.4.** *Let  $K$  be a finite oriented simplicial complex, and let  $d$  be an integer with  $0 \leq d \leq \dim(K)$ . Let  $\sigma_1$  and  $\sigma_2$  be distinct and upper adjacent  $d$ -simplices of  $K$ , and let  $\tau$  be their common upper simplex. Then  $\sigma_1$  and  $\sigma_2$  are similarly oriented with respect to  $\tau$  iff they have a dissimilar reduced common lower simplex.*

**Proof.** If  $d > 0$ , the proof is the same as in the un-reduced case in Lemma 3.2.6. Suppose  $d = 0$ . We know that any two upper adjacent vertices are dissimilarly oriented with respect to their common upper simplex. Since  $\sigma_1$  and  $\sigma_2$  are vertices, we also know that they are reduced lower adjacent, and that the empty set is a similar reduced common lower simplex of them. Since two vertices cannot be similarly oriented with respect to a common upper simplex, this completes the proof.  $\square$

We again have the following corollary.

**Corollary 3.4.5.** *Let  $d \geq 0$  be an integer. If two distinct  $d$ -simplices of a finite simplicial complex are upper adjacent, then they are also reduced lower adjacent.*

Since all definitions for reduced Laplacians are identical to their un-reduced analogues in any dimension greater than 0, the following proposition is completely equivalent to its analogue in the un-reduced case, namely Proposition 3.3.2.

**Proposition 3.4.6.** *Let  $K$  be a finite oriented simplicial complex, and let  $d$  be an integer with  $0 \leq d \leq \dim(K)$ . Let  $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$  be the  $d$ -simplices of  $K$ . Let  $i, j \in \{1, 2, \dots, n\}$ . Then*

$$(\tilde{\mathcal{L}}_d^{UP})_{ij} = \begin{cases} \deg_U(\sigma_i), & \text{if } i = j \\ 1, & \text{if } i \neq j \text{ and } \sigma_i \text{ and } \sigma_j \text{ are upper adjacent} \\ & \text{and oriented similarly} \\ -1, & \text{if } i \neq j \text{ and } \sigma_i \text{ and } \sigma_j \text{ are upper adjacent} \\ & \text{and oriented dissimilarly} \\ 0, & \text{if } i \neq j \text{ and } \sigma_i \text{ and } \sigma_j \text{ are not upper adjacent.} \end{cases}$$

The proof of the next proposition is nearly the same as the proof of Proposition 3.3.3, except that here we can use one proof for all dimensions, instead of different proofs for dimension 0 and dimensions greater than 0.

**Proposition 3.4.7.** *Let  $K$  be a finite oriented simplicial complex, and let  $d$  be an integer with  $0 \leq d \leq \dim(K)$ . Let  $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$  be the  $d$ -simplices of  $K$ . Let  $i, j \in \{1, 2, \dots, n\}$ . Then*

$$(\tilde{\mathcal{L}}_d^{DN})_{ij} = \begin{cases} \deg_{\bar{L}}(\sigma_i), & \text{if } i = j \\ 1, & \text{if } i \neq j \text{ and } \sigma_i \text{ and } \sigma_j \text{ have a similar} \\ & \text{reduced common lower simplex} \\ -1, & \text{if } i \neq j \text{ and } \sigma_i \text{ and } \sigma_j \text{ have a dissimilar} \\ & \text{reduced common lower simplex} \\ 0, & \text{if } i \neq j \text{ and } \sigma_i \text{ and } \sigma_j \text{ are not reduced lower adjacent.} \end{cases}$$

**Proof.** If  $d > 0$ , then the proof is the same as the proof of its un-reduced analogue, Proposition 3.3.3. Suppose  $d = 0$ . Every vertex contains exactly one  $(-1)$ -face, the empty set, so the reduced lower degree of any vertex is 1. Any two distinct vertices are reduced lower adjacent, and the empty set is a similar reduced common lower simplex of them. From Remark 3.4.1, we know that  $\tilde{\mathcal{B}}_0^T \tilde{\mathcal{B}}_0$  is a matrix whose entries are all 1. This proves the proposition.  $\square$

Propositions 3.4.6 and 3.4.7 look very much like their unreduced counterparts, but the Reduced Laplacian Matrix Theorem is clearly slightly different, and essentially tidier, than its unreduced counterpart, the Laplacian Matrix Theorem.

**Theorem 3.4.8 (Reduced Laplacian Matrix Theorem).** *Let  $K$  be a finite oriented simplicial complex, and let  $d$  be an integer with  $0 \leq d \leq \dim(K)$ , and let  $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$  denote the  $d$ -simplices of  $K$ . Let  $i, j \in \{1, 2, \dots, n\}$ . Then*

$$(\tilde{\mathcal{L}}_d)_{ij} = \begin{cases} \deg_U(\sigma_i) + d + 1, & \text{if } i = j \\ 1, & \text{if } i \neq j \text{ and } \sigma_i \text{ and } \sigma_j \text{ are not upper adjacent} \\ & \text{but have a similar reduced common lower simplex} \\ -1, & \text{if } i \neq j \text{ and } \sigma_i \text{ and } \sigma_j \text{ are not upper adjacent} \\ & \text{but have a dissimilar reduced common lower simplex} \\ 0, & \text{if } i \neq j \text{ and either } \sigma_i \text{ and } \sigma_j \text{ are upper adjacent} \\ & \text{or are not reduced lower adjacent.} \end{cases}$$

**Proof.** The proof of this theorem is identical to the proof of part (ii) of this theorem's un-reduced analogue, Theorem 3.3.4, which proves the theorem if  $d > 0$ . The only reason the same argument did not apply previously if  $d = 0$  is that it relied on results that were not true for  $d = 0$ , namely Corollary 3.2.7 and Lemmas 3.2.6 and 3.3.3. However, in the reduced case, the analogues of these results, namely Corollary 3.4.5 and Lemmas 3.4.4 and 3.4.7, hold true for all integers  $d \geq 0$ .  $\square$

Again, we have the following Corollary. The proof of this result is identical to the proof of Corollary 3.3.7 (2).

**Corollary 3.4.9.** *Let  $K$  be a finite oriented simplicial complex. Let  $d$  be an integer with  $0 \leq d \leq \dim(K)$ , let  $\sigma_1, \dots, \sigma_n$  be the oriented  $d$ -simplices of  $K$ , and let  $i \in \{1, \dots, n\}$ . Then*

$$\tilde{\Delta}_d(\sigma_i) = \sum_{\sigma_j \sim_{\bar{L}} \sigma_i} (\sigma_i + s_{ij} \sigma_j) + \sum_{\sigma_k \sim_U \sigma_i} (\sigma_i - s_{ik} \sigma_k),$$

where  $s_{ij}$  is 1 if  $\sigma_i$  and  $\sigma_j$  have a similar reduced common lower simplex, and  $-1$  if they have a dissimilar reduced common lower simplex, for all  $i, j \in \{1, \dots, n\}$ .

# 4

## Laplacian Spectra of Simplicial Complexes

### 4.1 Spectra of $\Delta$ , $\tilde{\Delta}$ , $\Delta^{UP}$ , and $\Delta^{DN}$

We now present some extremely useful and pretty results about the eigenvalues of Laplacian operators.

**Definition.** Let  $K$  be a finite oriented simplicial complex, and let  $d$  be an integer with  $0 \leq d \leq \dim(K)$ . Then the  $d$ th **Laplacian spectrum** of  $K$ , denoted  $Spec(\Delta_d(K))$ , is the multiset of eigenvalues of  $\Delta_d$  of  $K$ .  $\triangle$

**Theorem 4.1.1.** *Let  $K$  be a finite simplicial complex, and let  $d$  be an integer with  $0 \leq d \leq \dim K$ . Then  $Spec(\Delta_d(K))$  is independent of the choice of orientation for the  $d$ -simplices of that complex.*

**Proof.** By Theorem 3.3.4, it is apparent that the diagonal entries of a Laplacian matrix are independent of orientation, and similarly we note that whether a nondiagonal entry is zero or nonzero is independent of orientation as well. So, we see that only the signs of the nonzero nondiagonal entries of the Laplacian matrix are dependent on orientation. If  $d = 0$ , then this is not an issue, since all nondiagonal entries of  $\mathcal{L}_0$  are either 0 or  $-1$  for any simplicial complex.

Suppose  $d > 0$ . Let  $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$  denote the  $d$ -simplices of  $K$ . Suppose  $K_1$  is the simplicial complex  $K$  with some arbitrary orientation given to the  $d$ -simplices, and that  $K_2$  is also the complex  $K$  with the same orientations on the  $d$ -simplices as in  $K_1$  except that  $\sigma_p$  has the opposite orientation in  $K_2$  as it does in  $K_1$ , for some  $p \in \{1, 2, \dots, n\}$ . Let  $\mathcal{L}_d(K_1)$  and  $\mathcal{L}_d(K_2)$  denote the  $d$ th Laplacian matrices of  $K_1$  and  $K_2$ , respectively. Let  $\sigma_p$  denote the elementary chain of this simplex with respect to its orientation given in  $K_1$ , and then  $-\sigma_p$  will denote the elementary chain of this same simplex with respect to its orientation given in  $K_2$ .

Suppose  $\sigma_i$  is lower but not upper adjacent with  $\sigma_p$ . Let  $\eta$  be the common lower simplex of  $\sigma_i$  and  $\sigma_p$ . Note that  $\partial_d(-\sigma_p) = -\partial_d(\sigma_p)$ , so the coefficient of  $\eta$  in  $\partial_d(-\sigma_p)$  has the

opposite sign from its sign in  $\partial_d(\sigma_p)$ . Hence, if the coefficient of  $\eta$  in  $\partial_d(\sigma_i)$  has the same sign as its coefficient in  $\partial_d(\sigma_p)$ , then the coefficient in  $\partial_d(\sigma_i)$  has the opposite sign as its coefficient in  $\partial_d(-\sigma_p)$ . Similarly, if the coefficient of  $\eta$  in  $\partial_d(\sigma_i)$  has the opposite sign as its coefficient in  $\partial_d(\sigma_p)$ , then the coefficient in  $\partial_d(\sigma_i)$  has the same sign as its coefficient in  $\partial_d(-\sigma_p)$ . Therefore if  $\sigma_p$  and  $\sigma_i$  have a similar common lower simplex in  $K_1$ , then they have a dissimilar common lower simplex in  $K_2$ , and vice versa. It follows that  $(\mathcal{L}_d(K_1))_{pi} = (\mathcal{L}_d(K_1))_{ip}$  has the opposite sign of  $(\mathcal{L}_d(K_2))_{pi} = (\mathcal{L}_d(K_2))_{ip}$ .

Since this holds true for any  $\sigma_i$  with which  $\sigma_p$  is lower but not upper adjacent, we see that  $\mathcal{L}_d(K_2)$  is the same as  $\mathcal{L}_d(K_1)$  except that every nondiagonal nonzero entry in row  $p$  or in column  $p$  has the opposite sign. This result can be obtained by multiplying each entry in row  $p$  by  $-1$  and then multiplying each entry in column  $p$  by  $-1$ , because this will not affect any zero entries, and the  $p$ th diagonal entry will be multiplied by  $-1$  twice, leaving it alone.

For each  $x \in \{1, 2, \dots, n\}$  let  $J_x$  denote the  $n \times n$  matrix given by

$$(J_x)_{ij} = \begin{cases} -1, & \text{if } i = x = j \\ 1, & \text{if } x \neq i = j \\ 0, & \text{otherwise.} \end{cases}$$

For all  $x \in \{1, 2, \dots, n\}$  note that  $J_x$  is invertible and is in fact idempotent. Also, note that for any  $n \times n$  matrix  $A$ , multiplication on the left by  $J_x$  will yield a matrix identical to  $A$  except with all the entries in row  $x$  multiplied by  $-1$ , and multiplication by  $J_x$  on the right yields a matrix identical to  $A$  except with all the entries in column  $x$  multiplied by  $-1$ .

We see then that  $\mathcal{L}_d(K_2) = J_p \mathcal{L}_d(K_1) J_p$ . Since  $J_p$  is idempotent, this means that  $\mathcal{L}_d(K_2)$  and  $\mathcal{L}_d(K_1)$  are similar matrices. A well-known result in linear algebra states that similar matrices have identical eigenvalues, (see [FIS97, page 240]), so it follows that  $\text{Spec}(\Delta_d(K_1)) = \text{Spec}(\Delta_d(K_2))$ .

The above argument dealt with the case where the orientation of simplices of a particular dimension in two geometrically identical finite simplicial complexes differed only in the orientation of a single simplex. Since we are dealing with finite simplicial complexes, the complete result of this theorem follows by a simple application of mathematical induction.  $\square$

**Example 4.1.2.** In Figure 4.1.1, we have two oriented simplicial complexes  $K_1$  and  $K_2$  that are geometrically identical but have different orientations on their simplices. We calculate that

$$\mathcal{L}_1(K_1) = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix} \text{ and } \mathcal{L}_1(K_2) = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 3 & 1 \\ 0 & -1 & 1 & 2 \end{pmatrix},$$

but  $\text{Spec}(\Delta_1(K_1)) = \{1, 3, 3, 4\} = \text{Spec}(\Delta_1(K_2))$ .  $\diamond$

**Definition.** Let  $K$  be a finite oriented simplicial complex, and let  $d$  be an integer with  $0 \leq d \leq \dim(K)$ . Then the  $d$ th **reduced Laplacian spectrum** of  $K$ , denoted  $\text{Spec}(\tilde{\Delta}_d(K))$ , is the multiset of eigenvalues of  $\tilde{\Delta}_d$  of  $K$ .  $\triangle$



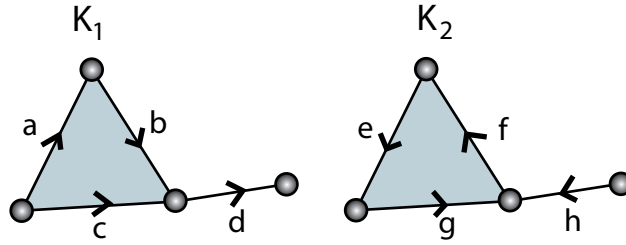


Figure 4.1.1.

**Theorem 4.1.3.** *Let  $K$  be a finite simplicial complex, and let  $d$  be an integer with  $0 \leq d \leq \dim K$ . Then  $\text{Spec}(\tilde{\Delta}_d(K))$  is independent of the choice of orientation for the  $d$ -simplices of that complex.*

**Proof.** If  $d > 0$ , then the  $d$ th reduced Laplacian matrix is the same as the  $d$ th Laplacian matrix, so the proof here is the same as that in Theorem 4.1.1 for  $d > 0$ . Suppose  $d = 0$ . From Remark 3.4.1, we know that each entry of the 0-dimensional reduced Laplacian matrix  $\tilde{\mathcal{L}}_0$  is 1 more than the corresponding entry in the 0-dimensional un-reduced Laplacian matrix  $\mathcal{L}_0$ . From Theorem 3.3.4, we note that the values of the entries of  $\mathcal{L}_0$  are independent of the choice of orientation of the vertices, precisely because there is no choice of orientation for vertices. It follows that the matrix  $\tilde{\mathcal{L}}_0$  is in fact independent of the choice of orientation of the vertices, so its eigenvalues are independent of this as well.  $\square$

**Remark.** In Remark 3.3.6 we noted that the value of the  $d$ th Laplacian matrix of a simplicial complex  $K$ , and clearly that of the  $d$ th reduced Laplacian matrix, depends only on the orientations of the  $d$ -simplices. We have just seen that the eigenvalues of these matrices do not depend on orientation at all!! This allows us to discuss the Laplacian spectra, or reduced Laplacian spectra, of a finite simplicial complex without specifying any orientations.  $\diamond$

The kernel of the Laplacian operator has a very fascinating and useful property. This property is called the Combinatorial Hodge Theorem. The Combinatorial Hodge Theorem is stated and proved in [YU83, Lemma 4], although there they use chains over  $\mathbb{Q}$  instead of  $\mathbb{R}$ . The proof for chains over  $\mathbb{R}$  is essentially the same. Note that this theorem requires that we work with chains over a field, and is not generally true for chains over an arbitrary group.

**Theorem 4.1.4 (Combinatorial Hodge Theorem for Laplacians).** *Let  $K$  be a finite simplicial complex. Then*

$$H_i(K; \mathbb{R}) \cong N(\Delta_i(K))$$

*for each integer  $i$  with  $0 \leq i \leq \dim(K)$ , where the isomorphism is as vector spaces over  $\mathbb{R}$ .*

It may seem that the presence of both reduced and unreduced Laplacians in zero dimensions means that we have another set of eigenvalues to take into account for each simplicial complex. Also, in [DURE], the definition of the Laplacian operator used by Duval and Reiner is actually the definition of what we call the reduced Laplacian, so it would

be nice if there is some easy way to convert between their Laplacian spectra and ours, even though reduced and unreduced Laplacians are only different in the case of dimension 0. The following result makes this connection.

**Theorem 4.1.5.** *Let  $K$  be a finite simplicial complex with  $n$  vertices. Then*

$$\text{Spec}_{NZ}(\tilde{\Delta}_0(K)) = \text{Spec}_{NZ}(\Delta_0(K)) \cup_M \{n\},$$

and

$$\text{Spec}(\tilde{\Delta}_d(K)) = \text{Spec}(\Delta_d(K))$$

for all integers  $d > 0$ .

**Proof.** The second equation in the statement of the theorem follows from the fact that for all integers  $d > 0$  we have  $\Delta_d(K) = \tilde{\Delta}_d(K)$ .

For the first equation, let  $\{v_1, v_2, \dots, v_n\}$  denote the vertices of  $K$ . Recall the definition of  $\mathbb{U}_n$  from Lemma 2.2.9. By Remark 3.4.1 we know that  $\tilde{\mathcal{L}}_0 = \mathcal{L}_0 + \mathbb{U}_n$ . We will show that  $\mathcal{L}_0\mathbb{U}_n$  and  $\mathbb{U}_n\mathcal{L}_0$  are both the  $n \times n$  zero matrix, so that  $\Delta_0$  and the linear operator  $C_0 \rightarrow C_0$  represented by  $\mathbb{U}_n$  are mutually annihilating operators.

Let  $i, j \in \{1, 2, \dots, n\}$ . Making use of Theorem 3.3.4, we see that the  $ij$ th component of  $\mathcal{L}_0\mathbb{U}_n$  is the sum of the components of the  $i$ th row of  $\mathcal{L}_0$ , which is

$$\sum_{j=1}^n (\mathcal{L}_0)_{ij} = \deg_U v_i + \sum_{v_k \sim_U v_i} (-1).$$

Since the upper degree of a vertex is precisely the number of vertices with which it is upper adjacent, we see that this sum and hence this component must be 0. Similarly, the  $ij$ th component of  $\mathbb{U}_n\mathcal{L}_0$  is the sum of the components of the  $j$ th row of  $\mathcal{L}_0$ , which is

$$\sum_{i=1}^n (\mathcal{L}_0)_{ij} = \deg_U v_j + \sum_{v_k \sim_U v_j} (-1),$$

which must also be 0.

Since  $\Delta_0$  and the operator given by  $\mathbb{U}_n$  are self-adjoint as well as mutually annihilating, it follows by Theorem 2.2.5 that  $\text{Spec}_{NZ}(\mathcal{L}_0 + \mathbb{U}_n) = \text{Spec}_{NZ}(\mathcal{L}_0) \cup_M \text{Spec}_{NZ}(\mathbb{U}_n)$ . From Lemma 2.2.9 we know that  $\text{Spec}_{NZ}(\mathbb{U}_n) = \{n\}$ . Therefore  $\text{Spec}_{NZ}(\tilde{\Delta}_0(K)) = \text{Spec}_{NZ}(\Delta_0(K)) \cup_M \{n\}$ .  $\square$

In the case of a connected complex, Theorem 4.1.5 can be made even more specific, describing the actual spectra instead of the non-zero spectra.

**Corollary 4.1.6.** *Let  $K$  be a connected finite simplicial complex with  $n$  vertices. Then*

$$\text{Spec}(\tilde{\Delta}_0(K)) = \text{Spec}_{NZ}(\Delta_0(K)) \cup_M \{n\}.$$

**Proof.** From Theorem 4.1.4, we know that the dimension of the nullspace of  $\Delta_0$ , which is the multiplicity of 0 as an eigenvalue of  $\Delta_0$ , is the dimension of the 0th homology group of  $K$ . It is well known [MUN84, Theorem 7.1] that the dimension of the 0th homology group of  $K$  is equal to the number of components of  $K$ , which in this case is 1. Therefore

0 is an eigenvalue of multiplicity 1 of  $\Delta_0(K)$ , so  $\Delta_0(K)$  has precisely  $n - 1$  nonzero eigenvalues. Since  $\tilde{\Delta}_0(K)$  can have only  $n$  eigenvalues, by Theorem 4.1.5 we see that  $\text{Spec}(\tilde{\Delta}_0(K)) = \text{Spec}_{NZ}(\Delta_0(K)) \cup_M \{n\}$ .  $\square$

The following lemma describes the very useful relationship between the spectrum of  $\Delta$ , and the spectra of  $\Delta^{UP}$  and  $\Delta^{DN}$ , in both non-reduced and reduced settings.

**Lemma 4.1.7.** *Let  $K$  be a finite oriented simplicial complex, and let  $d \in \mathbb{Z}$  be such that  $0 \leq d \leq \dim(K)$ . Then*

$$\text{Spec}_{NZ}(\Delta_d(K)) = \text{Spec}_{NZ}(\Delta_d^{UP}(K)) \cup_M \text{Spec}_{NZ}(\Delta_d^{DN}(K))$$

and

$$\text{Spec}_{NZ}(\tilde{\Delta}_d(K)) = \text{Spec}_{NZ}(\tilde{\Delta}_d^{UP}(K)) \cup_M \text{Spec}_{NZ}(\tilde{\Delta}_d^{DN}(K)).$$

**Proof.** As linear operators, we know that both  $\Delta_d^{UP}$  and  $\Delta_d^{DN}$  are self-adjoint. From [MUN84, Lemma 5.3] we know that the image of the boundary operator of a given dimension is contained in the nullspace of the boundary operator of the next lower dimension, and from [FIS97, Theorem 6.11 (c)] we know that the adjoint of the composition of two linear operators is the composition in the opposite direction of the adjoints of the two operators. Also, we see easily that the composition on either side of a linear operator with the zero map of the proper dimension is the zero map of the proper dimension, and that the adjoint of a zero map is another zero map. Taken altogether, these facts imply that

$$\begin{aligned} \Delta_d^{UP} \circ \Delta_d^{DN} &= (\partial_{d+1} \circ \partial_{d+1}^*) \circ (\partial_d^* \circ \partial_d) = \partial_{d+1} \circ (\partial_{d+1}^* \circ \partial_d^*) \circ \partial_d \\ &= \partial_{d+1} \circ (\partial_d \circ \partial_{d+1})^* \circ \partial_d = \partial_{d+1} \circ \mathbf{0}^* \circ \partial_d \\ &= \partial_{d+1} \circ \mathbf{0} \circ \partial_d = \mathbf{0}. \end{aligned}$$

In a similar fashion, we have

$$\begin{aligned} \Delta_d^{DN} \circ \Delta_d^{UP} &= (\partial_d^* \circ \partial_d) \circ (\partial_{d+1} \circ \partial_{d+1}^*) = \partial_d^* \circ (\partial_d \circ \partial_{d+1}) \circ \partial_{d+1}^* \\ &= \partial_d^* \circ \mathbf{0} \circ \partial_{d+1}^* = \mathbf{0}. \end{aligned}$$

Therefore  $\Delta_d^{UP}$  and  $\Delta_d^{DN}$  are self-adjoint, mutually annihilating operators. Similarly for  $\tilde{\Delta}_d^{UP}$  and  $\tilde{\Delta}_d^{DN}$ . The desired results follow directly from Theorem 2.2.5.  $\square$

This next lemma displays the relationship between the spectra of  $\Delta^{UP}$  and  $\Delta^{DN}$ , again in both nonreduced and reduced settings.

**Lemma 4.1.8.** *Let  $K$  be a finite simplicial complex. Then*

$$\text{Spec}_{NZ}(\Delta_{d-1}^{UP}(K)) = \text{Spec}_{NZ}(\Delta_d^{DN}(K))$$

and

$$\text{Spec}_{NZ}(\tilde{\Delta}_{d-1}^{UP}(K)) = \text{Spec}_{NZ}(\tilde{\Delta}_d^{DN}(K))$$

for all positive integers  $d \leq \dim(K)$ .

**Proof.** Since  $\Delta_{d-1}^{UP} = \partial_d \partial_d^*$  and  $\Delta_d^{DN} = \partial_d^* \partial_d$ , and  $\tilde{\Delta}_{d-1}^{UP} = \tilde{\partial}_d \tilde{\partial}_d^*$  and  $\tilde{\Delta}_d^{DN} = \tilde{\partial}_d^* \tilde{\partial}_d$ , these results follow immediately from Theorem 2.2.6.  $\square$

As remarked in [DURE, page 9], it is seen from Lemmas 4.1.7 and 4.1.8 that for a finite simplicial complex the information carried by the spectra of  $\Delta$ , the spectra of  $\Delta^{UP}$ , and the spectra of  $\Delta^{DN}$  is essentially the same. Hence, in studying the Laplacian spectra we can feel free to study the spectra of whichever of these three families of operators is most convenient.

In 2 dimensions, we have this corollary, which is simply a special case of Lemma 4.1.7.

**Corollary 4.1.9.** *Let  $K$  be a finite simplicial complex of dimension 2. Then*

$$\text{Spec}_{NZ}(\Delta_1(K)) = \text{Spec}_{NZ}(\Delta_0(K)) \cup_M \text{Spec}_{NZ}(\Delta_2(K)).$$

**Proof.** Note that  $\text{Spec}(\Delta_0(K)) = \text{Spec}(\Delta_0^{UP}(K))$  and  $\text{Spec}(\Delta_2(K)) = \text{Spec}(\Delta_2^{DN}(K))$ , because  $K$  is 2-dimensional. Then by Lemmas 4.1.7 and 4.1.8, respectively, we have  $\text{Spec}_{NZ}(\Delta_1(K)) = \text{Spec}_{NZ}(\Delta_1^{DN}(K)) \cup_M \text{Spec}_{NZ}(\Delta_1^{UP}(K)) = \text{Spec}_{NZ}(\Delta_0^{UP}(K)) \cup_M \text{Spec}_{NZ}(\Delta_2^{DN}(K)) = \text{Spec}_{NZ}(\Delta_0(K)) \cup_M \text{Spec}_{NZ}(\Delta_2(K))$ .  $\square$

## 4.2 Further Facts about Laplacian Spectra

Since the Laplacian matrix is the sum of compositions of linear operators with their adjoints, it is self-adjoint. By Theorem 6.17 and the lemma preceding it in [FIS97], we know that the Laplacian matrix is diagonalizable and that all of its eigenvalues are real. From Lemmas 2.2.8 and 2.2.7, we know that the Laplacian operator is positive semidefinite, and has only nonnegative eigenvalues. The following result goes even further, and presents some good news for people who do not like fractions, (or bad news for people who like them).

**Theorem 4.2.1.** *Let  $K$  be a finite simplicial complex, and let  $i$  be an integer with  $0 \leq i \leq \dim(K)$ . Then  $\text{Spec}(\Delta_i(K))$  contains no non-integer rational numbers.*

**Proof.** The eigenvalues of  $\Delta_i(K)$  are the roots of its characteristic polynomial  $f(t) = \det(\mathcal{L}_i(K) - tI_{f_i(K)})$ , where  $\mathcal{L}_i(K)$  is the matrix of  $\Delta_i(K)$  relative to an arbitrary ordering and orientation of the  $i$ -simplices of  $K$ , and  $I_{f_i(K)}$  is the  $f_i(K) \times f_i(K)$  identity matrix. We know by Theorem 3.3.4 that  $\mathcal{L}_i(K)$  is an integer matrix, so it is easy to deduce that  $f(t)$  is a polynomial over the integers. By [FIS97, Theorem 5.8], we know that the lead coefficient of  $f(t)$  is  $(-1)^{f_i(K)}$ . By the Rational Roots Theorem ([AMB77, Theorem 13.10]), we know that the denominator of any rational root of a polynomial over the integers must be a factor of the lead coefficient of that polynomial. Hence the only rational roots of  $f(t)$  have denominator divisible by  $\pm 1$ , so they are integers. Therefore the only roots of  $f(t)$  are integers or irrational numbers.  $\square$

**Example 4.2.2.** Even though Laplacian spectra do not contain fractions, they can definitely contain lots of irrational numbers. By the Laplacian Matrix Theorem, Theorem 3.3.4, we can calculate that for the simplicial complex  $K$  in Figure 4.2.1 we have that  $\text{Spec}(\Delta_0(K))$  consists of  $0, \frac{1}{2}(7 \pm \sqrt{13})$ , and the three roots of the cubic polynomial  $x^3 - 9x^2 + 23x - 14$ , which are approximately 0.885092, 3.2541, 4.86081. Also, we calculate that

$$\text{Spec}(\Delta_2(K)) = \{3, 3 \pm \sqrt{2}\}.$$

Therefore, the Laplacian spectra of this simplicial complex is almost completely irrational, although from its picture  $K$  seems like a most respectable and reasonable simplicial complex.

However, many of the simplicial complexes we look at do have integer Laplacian spectra. The main results from [DURE] proves that a large class of simplicial complexes, called shifted complexes do have integer Laplacian spectra.  $\diamond$

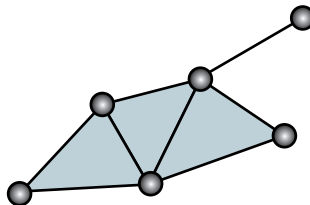


Figure 4.2.1.

Many of our results are about connected simplicial complexes. The following lemma tells us that dealing with the spectra of disconnected simplicial complexes amounts to dealing with the spectra of the complex's components.

**Lemma 4.2.3.** *Let  $K$  be a finite simplicial complex, and let its components be denoted  $K_1, K_2, \dots, K_n$ . Then*

$$\text{Spec}(\Delta_d(K)) = \text{Spec}(\Delta_d(K_1)) \cup_M \text{Spec}(\Delta_d(K_2)) \cup_M \dots \cup_M \text{Spec}(\Delta_d(K_n))$$

for all integers  $d$  with  $0 \leq d \leq \dim K$ .

**Proof.** Choose some ordering of the  $d$ -simplices of  $K$  such that the simplices of  $K_1$  are listed first, followed by the simplices of  $K_2$ , and so on until the simplices of  $K_n$ . It is clear that if two  $d$ -simplices of  $K$  are in two different components of  $K$ , then they can be neither upper nor lower adjacent. By similar reasoning, the upper and lower degrees of a  $d$ -simplex of  $K$  must be equal to the upper and lower degrees of that simplex, respectively, in the component of  $K$  containing the simplex. Therefore by Theorem 3.3.4 we have

$$\mathcal{L}_d(K) = \begin{pmatrix} \mathcal{L}_d(K_1) & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathcal{L}_d(K_2) & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathcal{L}_d(K_n) \end{pmatrix}$$

where the Laplacian matrices for  $K_1, K_2, \dots, K_n$  are relative to orderings of the  $d$ -simplices of these components given by the ordering of all the  $d$ -simplices of  $K$ , and where  $\mathbf{0}$  in each case denotes the zero matrix with appropriate dimensions. Therefore Lemma 2.2.10 implies the desired result.  $\square$

Although gluing two simplicial complexes together along a single simplex of both complexes affects the Laplacian spectrum of the complex in some dimensions, it turns out that some spectra are essentially undisturbed by the gluing.

**Lemma 4.2.4.** *Let  $K_1$  and  $K_2$  be finite simplicial complexes, and let  $K$  be a simplicial complex formed by gluing  $K_1$  and  $K_2$  together along a  $d$ -simplex of each. Then*

$$\text{Spec}(\Delta_i(K)) = \text{Spec}(\Delta_i(K_1)) \cup_M \text{Spec}(\Delta_i(K_2))$$

for all  $i \geq d + 2$ .

**Proof.** Let  $i \geq d + 2$  be an integer. By Theorem 3.3.4, we know that the entries of the  $i$ th Laplacian matrix are dependent only on upper and lower adjacencies and degrees, and we know that this is dependent only on the relationships between  $(i - 1)$ ,  $i$ , and  $i + 1$ -simplices, and not the simplices of any lower dimension than  $i - 1$ . Note that the  $i$ -simplices of  $K$  are exactly the  $i$ -simplices of  $K_1$  and  $K_2$ ; i.e. no  $i$  simplices are lost nor gained in the gluing.

Choose an ordering of the  $i$ -simplices of  $K$  such that the simplices of  $K_1$  come before the simplices of  $K_2$ . By what we stated above, it is clear that the relations between  $i$ -simplices of  $K$  originally contained in  $K_1$  are the same as they were in  $K_1$ , and similarly for  $K_2$ . Furthermore, if two  $i$ -simplices of  $K$  did not both come from either  $K_1$  or  $K_2$ , then they can be neither upper nor lower adjacent. By similar reasoning, the upper and lower degrees of an  $i$ -simplex of  $K$  must be equal to the upper and lower degrees of that simplex, respectively, in whichever of  $K_1$  or  $K_2$  originally contained the simplex. Therefore, we conclude that

$$\mathcal{L}_d(K) = \begin{pmatrix} \mathcal{L}_d(K_1) & \mathbf{0} \\ \mathbf{0} & \mathcal{L}_d(K_2) \end{pmatrix}$$

and as in Lemma 4.2.3, this proves the lemma.  $\square$

**Example 4.2.5.** Figure 4.2.2 shows two simplicial complexes being glued together along a vertex. By Lemma 4.2.4, the 2-dimensional Laplacian spectrum of the complex on the right is the multiset union of the 2-dimensional Laplacian spectra of the two complexes on the left.  $\diamond$

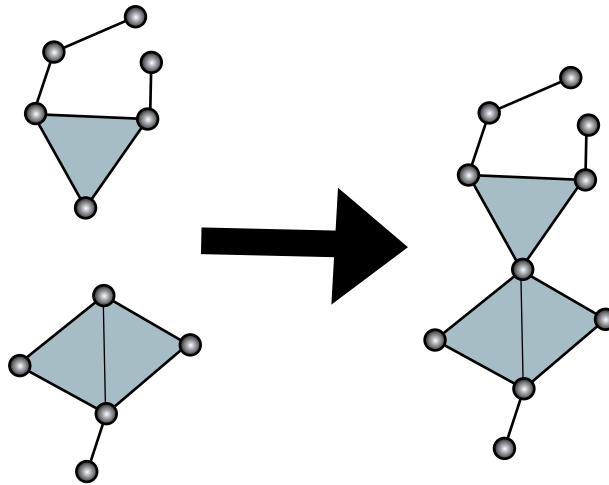


Figure 4.2.2.

Finally, we state the following result without proof.

**Lemma 4.2.6.** *Let  $K$  and  $K'$  be finite simplicial complexes, and let  $n$  be a positive integer such that the  $n$ -skeletons of  $K$  and  $K'$  are combinatorially equivalent. Then  $\text{Spec}(\Delta_i(K)) = \text{Spec}(\Delta_i(K'))$  for all nonnegative integers  $i < n$ .*

The proof of this lemma follows immediately from the Laplacian Matrix Theorem, Theorem 3.3.4, which states that the Laplacian spectrum in a given dimension depends only on the structures of simplices in that dimension and the dimension above it, but no simplices in dimensions higher than that.

Lemma 4.2.6 generalizes the following intuitive notion. If we start with a 1-dimensional simplicial complex, we can “fill in” any boundary of a triangle in our complex with a 2-simplex, and not affect the 0 Laplacian spectrum.

**Example 4.2.7.** The two simplicial complexes in Figure 4.2.3 have the same 1-skeleton, so by Lemma 4.2.6 they have the same Laplacian spectra in dimension 0.  $\diamond$

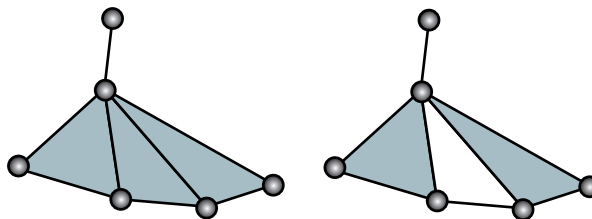


Figure 4.2.3.

# 5

## Laplacian Spectra of Specific Complexes

### 5.1 Flapoid Clusters, Cliques, and Graphs

This section presents results about the 0th Laplacian spectrum of simplicial complexes, which is essentially the spectrum of the graph theory Laplacian.

In graph theory, a graph consisting of  $n$  vertices with an edge between every pair of distinct vertices is called a complete graph on  $n$  vertices. It is also sometimes called an  $n$ -**clique**, which is how we shall refer to it here. The motivation behind the name clique is that the vertices of a graph can be thought of as representing people, with an edge between two vertices if the people represented by these vertices are friends. (See Figure 5.1.1.) Hence a clique represents a set of people each pair of whom are friends.

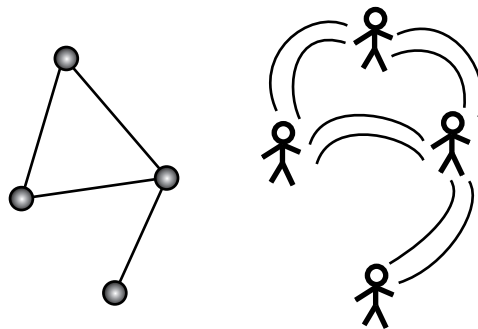


Figure 5.1.1.

**Example 5.1.1.** Figure 5.1.2 displays a clique on 5 vertices, not a satanic symbol of power.  $\diamond$

The Laplacian spectrum of a clique is given in the following result.



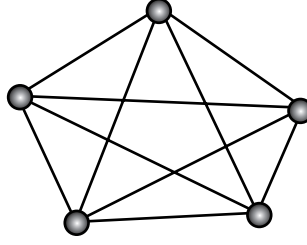


Figure 5.1.2.

**Proposition 5.1.2.** *Let  $n$  be a positive integer, and let  $G$  be an  $n$ -clique. Then*

$$\text{Spec}(\Delta_0(G)) = \{0, [n]^{n-1}\}.$$

**Proof.** Since every vertex in  $G$  has upper degree  $n - 1$  and every pair of distinct vertices in  $G$  are upper adjacent to each other, the Theorem 3.3.4 implies that

$$\mathcal{L}_0(G) = \begin{pmatrix} n-1 & -1 & \dots & -1 \\ -1 & n-1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & n-1 \end{pmatrix}.$$

By Remark 3.4.1 we deduce that

$$\tilde{\mathcal{L}}_0(G) = \begin{pmatrix} n & 0 & \dots & 0 \\ 0 & n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & n \end{pmatrix},$$

so clearly  $\text{Spec}(\tilde{\Delta}_0(G)) = \{[n]^n\}$ . Since  $G$  is connected, Corollary 4.1.6 implies that  $\text{Spec}(\Delta_0(G)) = \{0, [n]^{n-1}\}$ .  $\square$

Keeping with the idea of vertices representing people and edges representing friendships, we say that a vertex  $v$  of a finite simplicial complex  $K$  is **popular** if  $\deg_U(v) = f_0(K) - 1$ ; that is, if the person represented by this vertex is friends with everyone else represented by vertices of the complex.

**Example 5.1.3.** In Figure 5.1.3, the vertex  $v$  is popular, but that doesn't necessarily mean it's happy.  $\diamond$

The presence of popular vertices in a simplicial complex has a very strong influence on the 0th Laplacian spectra of that complex.

**Lemma 5.1.4.** *Let  $K$  be a finite simplicial complex. Suppose there are vertices  $v_1, \dots, v_m$  in  $K$  all of whom are popular, with  $0 < m < f_0(K)$ . Then*

$$[f_0(K)]^m \in \text{Spec}(\Delta_0(K)).$$

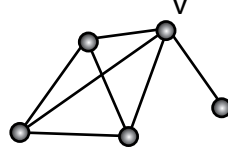


Figure 5.1.3.

**Proof.** Let  $n = f_0(K) > 0$ . Let  $v_1, v_2, \dots, v_m, \dots, v_n$  denote the vertices of  $K$ . For each  $i \in \{1, \dots, m\}$  let  $\vec{x}_i$  be the  $n$ -dimensional vector whose  $i$ th entry is  $n - 1$  and whose other entries are all  $-1$ .

Let  $i \in \{1, \dots, m\}$ . We will compute the  $i$ th entry of  $\mathcal{L}_0(K)\vec{x}_i$ . Since  $v_i$  is upper adjacent to all of the other vertices of  $K$ , by Theorem 3.3.4 the nondiagonal entries of the  $i$ th row of  $\mathcal{L}_0(K)$  are all  $-1$ . Hence the  $i$ th entry of  $\mathcal{L}_0(K)\vec{x}_i$  is  $(n - 1)\deg_U(v_i) + (n - 1) = n(n - 1)$ .

Now we will compute the other entries of  $\mathcal{L}_0(K)\vec{x}_i$ . Let  $j \in \{1, 2, \dots, n\} - \{i\}$ . Using Theorem 3.3.4, the only nonzero entries of the  $j$ th row of  $\mathcal{L}_0(K)$  are the  $j$ th entry, which is  $\deg_U(v_j)$ , and those corresponding to vertices to which  $v_j$  is upper adjacent, which are  $-1$ . One of these vertices must be  $v_i$ , leaving precisely  $\deg_U(v_j) - 1$  other vertices upper adjacent to  $v_j$ . From this information, we conclude that the  $j$ th entry of  $\mathcal{L}_0(K)\vec{x}_i$  is

$$(-1)(n - 1) + (-1)(-1)(\deg_U(v_j) - 1) - \deg_U(v_j) = -n.$$

We see then that  $\mathcal{L}_0(K)\vec{x}_i = n\vec{x}_i$ , and since  $\vec{x}_i$  is nonzero it is an eigenvector of  $\mathcal{L}_0(K)$  associated with the eigenvalue  $n$ .

To be sure of the multiplicity of  $n$ , we must check that  $x_1, \dots, x_m$  are linearly independent. Suppose  $a_1, \dots, a_m \in \mathbb{R}$  are such that  $a_1x_1 + \dots + a_mx_m = \vec{0}$ . Let  $i \in \{1, \dots, m - 1\}$ . From the  $i$ th and  $(i + 1)$ st rows of this vector equation we obtain the equations

$$-(n - 1)a_i + a_1 + \dots + a_{i-1} + a_{i+1} + \dots + a_m = 0$$

and

$$-(n - 1)a_{i+1} + a_1 + \dots + a_i + a_{i+2} + \dots + a_m = 0.$$

Subtracting the second equation from the first, we find that  $-na_i + na_{i+1} = 0$ , and since  $n \neq 0$  we have  $a_i = a_{i+1}$ . Since this holds for arbitrary  $i \in \{1, \dots, m - 1\}$ , we have  $a_1 = a_2 = \dots = a_m$ . Since  $m < n$ , we see that the last row of the above vector equation yields the equation  $a_1 + a_2 + \dots + a_m = 0$ . Hence  $ma_1 = 0$  so  $a_1 = a_2 = \dots = a_m = 0$ . Therefore the vectors  $x_1, \dots, x_m$  are linearly independent.  $\square$

We now have a second proof of our result on the Laplacian spectrum of a clique.

**Second Proof of Proposition 5.1.2.** We have an  $n$ -clique  $G$ . Note that each of the vertices of  $G$  is popular, so Lemma 5.1.4 guarantees that  $[n]^{n-1} \in \text{Spec}(\Delta_0(G))$ . By Theorem 4.1.4, since  $G$  is connected, we also have  $0 \in \text{Spec}(\Delta_0(G))$ . This accounts for all  $n$  eigenvalues of  $\Delta_0(G)$ , so  $\text{Spec}(\Delta_0(G)) = \{0, [n]^{n-1}\}$ .  $\square$

When a simplicial complex contains a popular vertex, the following conjecture seems to be true, although no proof has yet been found by the author.

**Conjecture 5.1.5.** *Let  $K$  be a finite simplicial complex containing a popular vertex. Then  $f_0(K)$  is the largest element of  $\text{Spec}(\Delta_0(K))$ .*

We will now investigate patterns in the Laplacian spectra in a class of 2-dimensional simplicial complexes called flapwheels. Intuitively, a flapwheel is a single edge with some number of triangular flaps hanging from it.

**Definition.** Let  $n$  be a positive integer. An  $n$ -**flapwheel** is a simplicial complex consisting of  $n$  distinct 2-simplices, called **flaps**, whose intersection is a single 1-simplex, called the **axis**. The two vertices of the axis are called **axial vertices**, and the vertices of the flapwheel not contained in the axis are called **flap vertices**.  $\triangle$

Figure 5.1.4 depicts a 5-flapwheel.

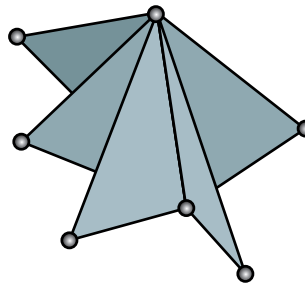


Figure 5.1.4.

We see easily that for all positive integers  $n$ , any  $n$ -flapwheel has exactly  $n + 2$  vertices and  $2n + 1$  edges. The following theorem completely categorizes the Laplacian spectra of flapwheels. We will present its proof shortly, after developing a little more machinery.

**Theorem 5.1.6.** *Let  $F$  be an  $n$ -flapwheel, where  $n$  is some positive integer. Then  $\text{Spec}(\Delta_0(F)) = \{0, [2]^{n-1}, n + 2, n + 2\}$  and  $\text{Spec}(\Delta_1(F)) = \{[2]^{2n-2}, n + 2, n + 2, n + 2\}$  and  $\text{Spec}(\Delta_2(F)) = \{[2]^{n-1}, n + 2\}$ .*

The following structures derive their name from their similarity to the flaps of flapwheels. In a general simplicial complex, a flap looks like a triangle glued onto the complex along a single edge. Often there are several flaps glued onto the complex on the same edge. In the 2-dimensional simplicial complex of Figure 5.1.5, the three left-most vertices are flap vertices of three flaps in the complex.

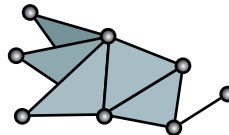


Figure 5.1.5.

The structures described below are a broad generalization of flaps, but they were first noticed by the author in the case of flaps, and are named as they are for this reason.

**Definition.** Let  $K$  be a simplicial complex, and let  $V$  be the set of vertices of  $K$ . The **vertex neighborhood** of a vertex  $v \in V$  is the set  $VNeigh(v) = \{x \in V \mid x \sim_U v\}$ .

A **flapoid cluster** is a subset  $F = \{v_1, v_2, \dots, v_n\} \subseteq V$  of vertices of  $K$  such that  $n \geq 2$ , the vertices in  $F$  are pairwise non-upper adjacent, and

$$VNeigh(v_1) = VNeigh(v_2) = \dots = VNeigh(v_n).$$

The set of vertices in the common vertex neighborhood of the vertices of a flapoid cluster  $F$  is called the **community** of  $F$ , and is denoted  $Com(F)$ .  $\triangle$

**Example 5.1.7.** In Figure 5.1.6, vertices 1 through 3 form a flapoid cluster, and their community is vertices 4 through 7.  $\diamond$

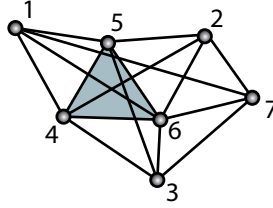


Figure 5.1.6.

The motivation for defining flapoid clusters is to state the following result about the zero Laplacian spectrum.

**Theorem 5.1.8.** *Let  $K$  be a finite simplicial complex, and suppose  $K$  contains distinct flapoid clusters  $F_1, F_2, \dots, F_k$  with  $|Com(F_1)| = |Com(F_2)| = \dots = |Com(F_k)| = d$ . Then*

$$[d]^{|F_1|+|F_2|+\dots+|F_k|-k} \in Spec(\Delta_0(K)).$$

**Proof.** Let  $n$  be the number of vertices in  $K$ . For each  $i \in \{1, 2, \dots, k\}$ , let  $n_i = |F_i|$  and let  $v_{i1}, v_{i2}, \dots, v_{in_i}$  denote the vertices in  $F_i$ . Choose some ordering  $L$  of the vertices of  $K$  such that  $L$  begins  $v_{11}, v_{12}, \dots, v_{1n_1}, v_{21}, v_{22}, \dots, v_{2n_2}, \dots, v_{k1}, v_{k2}, \dots, v_{kn_k}$ . For each  $i \in \{1, 2, \dots, k\}$  and  $j \in \{1, 2, \dots, n_i\}$ , let  $L(v_{ij})$  be the integer such that  $v_{ij}$  is the  $L(v_{ij})$ th entry in the ordering  $L$ .

Let  $i \in \{1, 2, \dots, k\}$ . Recall that by the definition of flapoid clusters, we have  $n_i > 1$ . For each  $j \in \{2, \dots, n_i\}$  let  $\vec{w}_{ij}$  be the vector whose  $L(v_{i1})$ th entry is  $-1$ , whose  $L(v_{ij})$ th entry is  $1$ , and all of whose other entries are  $0$ . Let  $V_i = \{\vec{w}_{i2}, \vec{w}_{i3}, \dots, \vec{w}_{in_i}\}$ .

Let  $j \in \{2, \dots, n_i\}$ . We will compute  $\mathcal{L}_0(K)\vec{w}_{ij}$ . Note that in making this computation, since only two entries of  $\vec{w}_{ij}$  are nonzero, we need only consider the  $L(v_{i1})$ th and the  $L(v_{ij})$ th columns of  $\mathcal{L}_0(K)$ . Furthermore, since  $v_{ij}$  is adjacent only to the vertices in  $Com(F_i)$ , by Theorem 3.3.4 the only nonzero entries of these two columns are the entries corresponding to the vertices in  $Com(F_i)$  and  $v_{i1}$  and  $v_{ij}$ .

Since each of the vertices in  $Com(F_i)$  are upper adjacent to both  $v_{i1}$  and  $v_{ij}$ , we see that the entries of  $\mathcal{L}_0(K)\vec{w}_{ij}$  corresponding to these vertices are all  $(-1 \times -1) + (-1 \times 1) = 1 - 1 = 0$ . The entries of the  $L(v_{i1})$ th and the  $L(v_{ij})$ th columns of  $\mathcal{L}_0(K)$  corresponding to  $v_{i1}$  and  $v_{ij}$  are diagonal entries, and therefore by Theorem 3.3.4 are equal to the upper degrees of

$v_{i1}$  and  $v_{ij}$ , respectively, both of which are  $d$  by the definition of flapoid clusters. We see then that the  $L(v_{i1})$ th and the  $L(v_{ij})$ th entries of  $\mathcal{L}_0(K)\vec{w}_{ij}$  are  $-d$  and  $d$ , respectively. Since all the other entries of  $\mathcal{L}_0(K)\vec{w}_{ij}$  are 0, we see then that  $\mathcal{L}_0(K)\vec{w}_{ij} = d\vec{w}_{ij}$ , so  $\vec{w}_{ij}$  is an eigenvector of  $\mathcal{L}_0(K)$  associated with  $d$ .

It is immediately clear that the  $n_i - 1$  vectors in  $V_i$  are linearly independent, and furthermore that the vectors in  $V_1 \cup V_2 \cup \dots \cup V_k$  are linearly independent, since the vertices of  $F_1, F_2, \dots, F_k$  are distinct. Since  $|V_1 \cup V_2 \cup \dots \cup V_k| = \sum_{i=1}^k (|F_i| - 1) = |F_1| + |F_2| + \dots + |F_k| - k$ , this means that the dimension of  $E_d(\mathcal{L}_0(K))$ , the eigenspace associated with the eigenvalue  $d$ , satisfies  $\dim(E_d(\mathcal{L}_0(K))) \geq |F_1| + |F_2| + \dots + |F_k| - k$ .  $\square$

**Example 5.1.9.** If we examine the case of flapoid clusters with community of size 2, we see that this is like a set of flaps with a common edge, except that we do not require that the flaps be triangles, or that they even have an edge in common! In a sense, all we require is the outline of the flaps.

For instance, let  $G$  be the simplicial complex pictured in Figure 5.1.7. This is a polygonal circle on four vertices, sometimes banally referred to as the boundary of a square. Each pair of opposite vertices forms a flapoid with two elements, and each flapoid has community of size 2, so Theorem 5.1.8 implies that  $\{2, 2\} \subseteq \text{Spec}(\Delta_0(G))$ . Furthermore, since  $G$  has only one component, by Theorem 4.1.4 we know that 0 is contained in  $G$ 's zero spectrum as well. Allowing ourselves to pluck a result from several pages ahead before its time, we see that  $G$  satisfies the hypotheses of Lemma 5.1.12, so the zero spectrum of  $G$  contains  $f_0(G) = 4$ . Therefore  $\text{Spec}(\Delta_0(G)) = \{0, 2, 2, 4\}$ . A quick calculation confirms this result.  $\diamond$

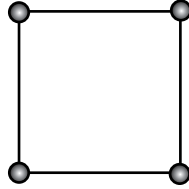


Figure 5.1.7.

We are now prepared to prove Theorem 5.1.6.

**Proof of Theorem 5.1.6.** The  $n$  flap vertices of the flapwheel form a flapoid cluster whose common community is the two axial vertices. Therefore by Theorem 5.1.8 we have  $[2]^{n-1} \in \text{Spec}(\Delta_0(F))$ . The two axial vertices of  $F$  are popular, so by Lemma 5.1.4, since  $2 < n + 2$ , we have  $[n + 2]^2 \in \text{Spec}(\Delta_0(F))$ . Finally, because  $F$  is connected, we have  $0 \in \text{Spec}(\Delta_0(F))$ . This accounts for all of  $\text{Spec}(\Delta_0(F))$ .

Now we investigate  $\text{Spec}(\Delta_2(F))$ . We give the 2-simplices of  $F$  some arbitrary ordering. Since there are no 3-simplices in  $F$ , the upper degree of each 2-simplex in  $F$  is 0. Note that every pair of 2-simplices share a lower simplex, and it is clear that we can orient the 2-simplices so that the axis of  $F$  is a similar common lower simplex of each pair of flaps. (If  $a$  and  $b$  are the axial vertices of  $F$ , for each flap vertex  $f$  let the 2-simplex that forms

this flap have the orientation  $\langle f, a, b \rangle$ .) By Theorem 3.3.4, we have

$$\mathcal{L}_2 = \begin{pmatrix} 3 & 1 & \dots & 1 \\ 1 & 3 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 3 \end{pmatrix}.$$

Again, let  $\vec{u}$  denote the  $n$ -dimensional column vector whose components are all 1. For each  $i \in \{1, 2, \dots, n-1\}$ , let  $\vec{v}_i$  denote the  $n$ -dimensional column vector whose coordinates are all 0 except that the last coordinate is  $-1$  and the  $i$ th coordinate is 1. We will demonstrate that  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{n-1}, \vec{u}\}$  is a basis for  $C_2$  consisting of eigenvectors of  $\mathcal{L}_2$ . (Similarly to before, if  $n = 1$ , then we ignore  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{n-1}$ . In what follows, note that the same arguments essentially hold if  $n = 1$  and we have no such vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{n-1}$ .)

Suppose there are  $k_1, k_2, \dots, k_n \in \mathbb{R}$  such that

$$k_1 \vec{v}_1 + k_2 \vec{v}_2 + \dots + k_{n-1} \vec{v}_{n-1} + k_n \vec{u} = \vec{0}.$$

By the definitions of these vectors, it follows that  $k_i + k_n = 0$  for all  $i \in \{1, 2, \dots, n-1\}$ , and also that  $-k_1 - k_2 - \dots - k_{n-1} + k_n = 0$ . From this first equation, it follows that  $k_n = -k_i$  for all  $i \in \{1, 2, \dots, n-1\}$ , so the second equation becomes  $(n-1)k_n + k_n = 0$ , so  $nk_n = 0$ . Since  $n \neq 0$ , this implies that  $k_n = 0$ , so  $k_1 = k_2 = \dots = k_{n-1} = 0$ . Therefore  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{n-1}, \vec{u}\}$  is linearly independent, and since this set contains  $n$  elements it must be a basis for  $C_2$ .

Now, note that

$$\mathcal{L}_2 \vec{u} = \begin{pmatrix} 3 & 1 & \dots & 1 \\ 1 & 3 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 3 + (n-1) \\ 3 + (n-1) \\ \vdots \\ 3 + (n-1) \end{pmatrix} = \begin{pmatrix} n+2 \\ n+2 \\ \vdots \\ n+2 \end{pmatrix} = (n+2)\vec{u},$$

so  $\vec{u} \in E_{n+2}(\mathcal{L}_2)$ . Observe also that

$$\mathcal{L}_2 \vec{v}_i = \begin{pmatrix} 3 & 1 & \dots & 1 \\ 1 & 3 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 3 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1-1 \\ \vdots \\ 3-1 \\ 1-1 \\ \vdots \\ 1-1 \\ 1-3 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 2 \\ 0 \\ \vdots \\ 0 \\ -2 \end{pmatrix} = 2\vec{v}_i$$

for all  $i \in \{1, 2, \dots, n-1\}$ , where the middle coordinate in each column vector in between the vertical dots is the  $i$ th coordinate, so  $\vec{v}_i \in E_2(\mathcal{L}_2)$ . Since  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{n-1}, \vec{u}\}$  is a basis for  $C_2$ , it follows that  $\text{Spec}(\Delta_2(F)) = \{n+2, [2]^{n-1}\}$ .

The statement about  $\text{Spec}(\Delta_1(F))$  follows immediately from Corollary 4.1.9 and our results about  $\text{Spec}(\Delta_0(F))$  and  $\text{Spec}(\Delta_2(F))$ .  $\square$

The following definition describes a situation related to that of cliques, except here you have a subset of vertices in a simplicial complex who have exactly the same friends and who are all friends with each other.

**Definition.** Let  $K$  be a simplicial complex, and let  $V$  be the set of vertices of  $K$ . A **neighborly clique** is a subset  $C = \{v_1, v_2, \dots, v_n\} \subseteq V$  of vertices of  $K$  such that  $n \geq 2$  and  $V\text{Neigh}(v_i) = \text{Com}(C) \cup (C - \{v_i\})$  for each  $i \in \{1, 2, \dots, n\}$ , where  $\text{Com}(C) = \bigcap_{i=1}^n V\text{Neigh}(v_i)$  is the **community** of  $C$ .  $\triangle$

**Example 5.1.10.** In Figure 5.1.8, vertices 1 through 3 form a neighborly clique, and their community is vertices 4 through 7.  $\diamond$

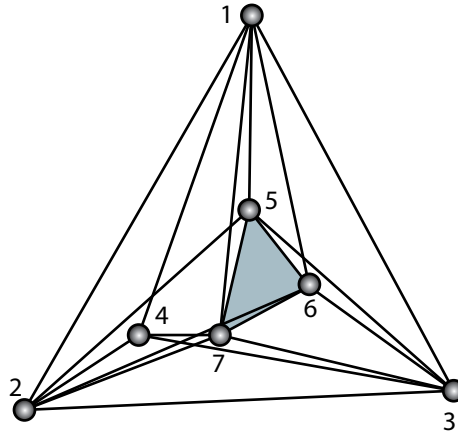


Figure 5.1.8.

It is clear from their definitions that flapoid clusters and neighborly cliques are similar in some way. Keeping with the metaphor behind the term “clique”, we can think of a flapoid clusters as representing the situation where a set of people are all friends with the same people, but are not friends with each other. There are then two ways to compare flapoid clusters and neighborly cliques. We can think of a neighborly clique as a flapoid cluster except that all of the vertices in the flapoid cluster are upper adjacent to each other. Another way we could think of a neighborly clique is as a flapoid cluster except with cliques instead of individual vertices. Whatever the mental picture in mind, we find that neighborly cliques are enough like flapoid clusters that we can obtain a similar result about the zero Laplacian spectrum.

**Theorem 5.1.11.** Let  $K$  be a finite simplicial complex, and suppose  $K$  contains distinct neighborly cliques  $C_1, C_2, \dots, C_k$  with  $|C_1 \cup \text{Com}(C_1)| = |C_2 \cup \text{Com}(C_2)| = \dots = |C_k \cup \text{Com}(C_k)| = d$ . Then

$$[d]^{|C_1|+|C_2|+\dots+|C_k|-k} \in \text{Spec}(\Delta_0(K)).$$

**Proof.** Let  $n$  be the number of vertices in  $K$ . For each  $i \in \{1, 2, \dots, k\}$ , let  $n_i = |C_i|$  and let  $v_{i1}, v_{i2}, \dots, v_{in_i}$  denote the vertices in  $C_i$ . Choose some ordering  $L$  of the vertices of  $K$  such that  $L$  begins  $v_{11}, v_{12}, \dots, v_{1n_1}, v_{21}, v_{22}, \dots, v_{2n_2}, \dots, v_{k1}, v_{k2}, \dots, v_{kn_k}$ . For each

$i \in \{1, 2, \dots, k\}$  and  $j \in \{1, 2, \dots, n_i\}$ , let  $L(v_{ij})$  be the integer such that  $v_{ij}$  is the  $L(v_{ij})$ th entry in the ordering  $L$ .

Let  $i \in \{1, 2, \dots, k\}$ . Recall that by the definition of neighborly cliques we have  $n_i > 1$ . For each  $j \in \{2, \dots, n_i\}$  let  $\vec{w}_{ij}$  be the vector whose  $L(v_{i1})$ th entry is  $-1$ , whose  $L(v_{ij})$ th entry is  $1$ , and all of whose other entries are  $0$ . Let  $V_i = \{\vec{w}_{i2}, \vec{w}_{i3}, \dots, \vec{w}_{in_i}\}$ .

Let  $j \in \{2, \dots, n_i\}$ . We will compute  $\mathcal{L}_0(K)\vec{w}_{ij}$ . Note that in making this computation, since only two entries of  $\vec{w}_{ij}$  are nonzero, we need only consider the  $L(v_{i1})$ th and the  $L(v_{ij})$ th columns of  $\mathcal{L}_0(K)$ . Furthermore, since  $v_{ij}$  is adjacent only to the vertices in  $C - \{v_i\}$  and  $Com(C_i)$ , by Theorem 3.3.4 the only nonzero entries of these two columns are the entries corresponding to the vertices in  $C_i$  and  $Com(C_i)$  and  $v_{i1}$  and  $v_{ij}$ .

Since every vertex in  $C_i - \{v_{i1}, v_{ij}\}$  and  $Com(C_i)$  is upper adjacent to both  $v_{i1}$  and  $v_{ij}$ , we see that the entries of  $\mathcal{L}_0(K)\vec{w}_{ij}$  corresponding to these vertices are all  $(-1 \times -1) + (-1 \times 1) = 1 - 1 = 0$ . The entries of the  $L(v_{i1})$ th and the  $L(v_{ij})$ th columns of  $\mathcal{L}_0(K)$  corresponding to  $v_{i1}$  and  $v_{ij}$  are diagonal entries, and therefore by Theorem 3.3.4 are equal to the upper degrees of  $v_{i1}$  and  $v_{ij}$ , respectively, both of which are  $|C_i| - 1 + |Com(C_i)| = d - 1$  by the definition of neighborly cliques. Since  $v_{i1}$  and  $v_{ij}$  are upper adjacent, we note also that the  $L(v_{ij})$ th entry of the  $L(v_{i1})$ th column of  $\mathcal{L}_0(K)$  and the  $L(v_{i1})$ th entry of the  $L(v_{ij})$ th column of  $\mathcal{L}_0(K)$  are both  $-1$ . We see then that the  $L(v_{i1})$ th and the  $L(v_{ij})$ th entries of  $\mathcal{L}_0(K)\vec{w}_{ij}$  are  $-(d - 1) - 1 = -d$  and  $(d - 1) - (-1) = d$ , respectively. Since all the other entries of  $\mathcal{L}_0(K)\vec{w}_{ij}$  are  $0$ , we see then that  $\mathcal{L}_0(K)\vec{w}_{ij} = d\vec{w}_{ij}$ , so  $\vec{w}_{ij}$  is an eigenvector of  $\mathcal{L}_0(K)$  associated with  $d$ .

It is immediately clear that the  $n_i - 1$  vectors in  $V_i$  are linearly independent, and furthermore that the vectors in  $V_1 \cup V_2 \cup \dots \cup V_k$  are linearly independent, since the vertices of  $C_1, C_2, \dots, C_k$  are distinct. Since  $|V_1 \cup V_2 \cup \dots \cup V_k| = \sum_{i=1}^k (|C_i| - 1) = |C_1| + |C_2| + \dots + |C_k| - k$ , this means that the dimension of  $E_d(\mathcal{L}_0(K))$ , the eigenspace associated with the eigenvalue  $d$ , satisfies  $\dim(E_d(\mathcal{L}_0(K))) \geq |C_1| + |C_2| + \dots + |C_k| - k$ .  $\square$

We finish this section by characterizing the Laplacian spectra of one last family of graphs.

**Definition.** Let  $G$  be a finite simplicial complex of dimension 1. Suppose there is a partition  $X, Y$  of the vertices  $G$  such that  $X$  and  $Y$  are nonempty, every pair of vertices from  $X$  and  $Y$  is upper adjacent, and the vertices in  $X$  are pairwise non-upper adjacent, and the vertices in  $Y$  are pairwise non-upper adjacent. If  $|X| = m$  and  $|Y| = n$ , then  $G$  is called a **complete bipartite graph on  $m$  and  $n$  vertices**.  $\triangle$

In Figure 5.1.9, we have a complete bipartite graph on 4 and 3 vertices.

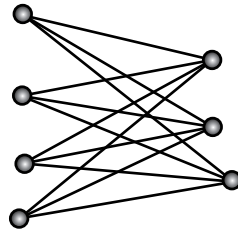


Figure 5.1.9.



**Lemma 5.1.12.** *Let  $K$  be a finite simplicial complex, and let  $V \subseteq K$  be the vertex set of  $K$ . Suppose there are disjoint and nonempty subsets  $X, Y \subseteq V$  such that  $V = X \cup Y$  and  $x$  is upper adjacent to  $y$  for all  $x \in X$  and  $y \in Y$ . Then*

$$f_0(K) \in \text{Spec}(\Delta_0(K)).$$

**Proof.** Let  $|X| = n$  and  $|Y| = m$ , and let  $x_1, \dots, x_n$  and  $y_1, \dots, y_m$  be the vertices of  $X$  and  $Y$ , respectively. We will use the ordering of  $V$  given by  $x_1, \dots, x_n, y_1, \dots, y_m$ . Since every vertex in  $X$  is upper adjacent to every vertex in  $Y$ , we see easily that we can view  $\mathcal{L}_0(K)$  as the block matrix

$$\mathcal{L}_0(K) = \begin{pmatrix} A & -\mathbf{1}_{n \times m} \\ -\mathbf{1}_{m \times n} & B \end{pmatrix},$$

where  $A$  and  $B$  are square matrices of dimensions  $n$  and  $m$ , respectively, and  $-\mathbf{1}_{n \times m}$  and  $-\mathbf{1}_{m \times n}$  are matrices all of whose entries are  $-1$  and whose dimensions are given by their subscripts.

Let  $\vec{v}$  denote the  $n + m$  dimensional vector whose  $i$ th component is  $-m$  if  $1 \leq i \leq n$ , and  $n$  if  $n < i \leq n + m$ . We will compute  $\mathcal{L}_0(K)\vec{v}$ .

Let  $i \in \{1, \dots, n\}$ . The  $i$ th entry of  $\mathcal{L}_0(K)\vec{v}$  is the dot product of the  $i$ th row of  $\mathcal{L}_0(K)$  and  $\vec{v}$ . We note that the  $i$ th diagonal entry of  $\mathcal{L}_0(K)$  will be multiplied by  $-m$  in this dot product, and that  $\deg_U(x_i)$  is exactly  $m$  plus the number of vertices in  $X$  to which  $x_i$  is upper adjacent. Using the Theorem 3.3.4, we calculate that the dot product determining the  $i$ th entry of  $\mathcal{L}_0(K)\vec{v}$  is  $-m \deg_U(x_i)$ , plus a term  $(-m)(-1)$  for each of the vertices in  $X$  to which  $x_i$  is upper adjacent, plus a term  $n(-1)$  for each of the vertices in  $Y$  to which  $x_i$  is upper adjacent; that is

$$-m \deg_U(x_i) + (-m)(-1)(\deg_U(x_i) - m) + (-1)nm = (n + m)(-m).$$

Since  $i$  was chosen arbitrarily, this implies that the first  $n$  entries of  $\mathcal{L}_0(K)\vec{v}$  are all  $(n + m)(-m)$ .

By completely similar reasoning, we see that for each  $j \in \{n + 1, \dots, n + m\}$ , we have that the  $j$ th entry of  $\mathcal{L}_0(K)\vec{v}$  is

$$(-1)(-m)n + n \deg_U(x_i) + n(-1)(\deg_U(x_i) - n) = (n + m)n.$$

Therefore  $\mathcal{L}_0(K)\vec{v} = (n + m)\vec{v} = f_0(K)\vec{v}$ . Since  $\vec{v} \neq \vec{0}$ , we see that  $\vec{v}$  is an eigenvector of  $\mathcal{L}_0(K)$  associated with the eigenvalue  $f_0(K)$ .  $\square$

**Proposition 5.1.13.** *Let  $m$  and  $n$  be positive integers, and let  $G$  be a complete bipartite graph on  $m$  and  $n$  vertices. Then*

$$\text{Spec}(\Delta_0(G)) = \{0, [m]^{n-1}, [n]^{m-1}, m + n\}.$$

**Proof.** Since  $G$  is connected, we know by Theorem 4.1.4 that  $0 \in \text{Spec}(\Delta_0(G))$ . Notice that  $X$  is a flapoid cluster whose community is  $Y$ , and that  $Y$  is a flapoid cluster whose community is  $X$ . By Theorem 5.1.8 we have  $[m]^{n-1}, [n]^{m-1} \in \text{Spec}(\Delta_0(G))$ . Finally, by Lemma 5.1.12 we have  $m + n = f_0(G) \in \text{Spec}(\Delta_0(G))$ . This accounts for all eigenvalues of  $\Delta_0(G)$ .  $\square$

## 5.2 Cones, Cones, and More Cones

Given any simplicial complex, we can form a new simplicial complex of one dimension higher by an operation called coning. For a formal definition of the cone of a simplicial complex see [MUN84, page 43]. Intuitively, we form the cone of a simplicial complex  $K$  by taking a point  $w$  completely separate from the complex and then forming new  $(d+1)$ -simplices by combining the point  $w$  with each  $d$ -simplex in  $K$ . We denote the cone of  $K$  with the point  $w$  by  $w * K$ , and we will call  $w$  the **coning vertex** of  $w * K$  and  $K$  the **base** of  $w * K$ . In Figure 5.2.1, we see the cone of the simplicial complex  $K$  with coning vertex  $w$ . In this picture, the front two triangles of the tetrahedron of  $w * K$  are left transparent, and all other simplices of  $w * K - K$  are shaded lighter than the shading of the simplices of  $K$ .

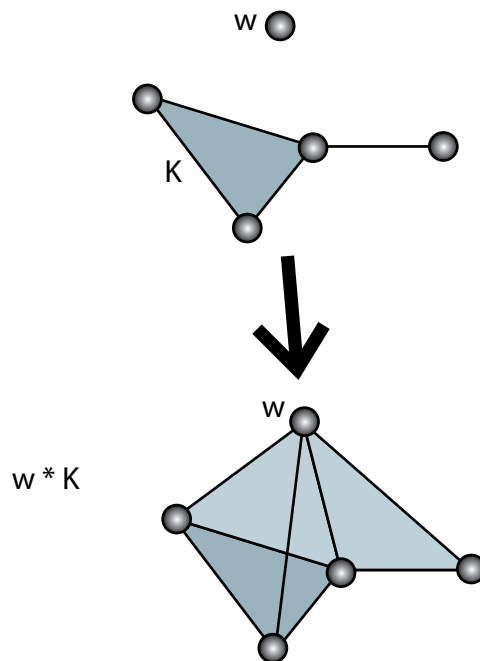


Figure 5.2.1.

A simplicial complex is a cone, sometimes called a simplicial cone, if it contains a subcomplex  $G$  and a vertex  $w$  such that  $K = w * G$ . Cones are extensively studied objects in topology and combinatorial algebra. The operation of coning a simplicial complex is a very interesting one combinatorially, because the cone of a simplicial complex is another simplicial complex, and often certain properties of the original complex are predictably altered by the process of coning. Such is the case to some extent for Laplacian spectra, as we will soon see. First we need a definition and lemma.

**Definition.** Let  $K$  be a finite oriented simplicial complex, let  $w$  be a vertex such that  $w * K$  is well-defined, and let  $d \in \mathbb{Z}$  be nonnegative. Each  $d$ -simplex  $\sigma \in (w * K) - K$  has exactly one face that is a  $(d-1)$ -simplex in  $K$ ; we call this  $(d-1)$ -simplex the **presimplex** of  $\sigma$ , and we denote it  $p(\sigma)$ . We define the presimplex of  $w$  to be  $p(w) = \emptyset$ .

Let  $\sigma \in (w * K) - K$  be a  $d$ -simplex. The orientation of  $p(\sigma)$  in  $K$  can be represented by a list of its vertices  $[v_1 \dots v_d]$ . The **standard orientation of  $\sigma$  induced by  $p(\sigma)$**  is the orientation given by

$$(-1)^d[v_1 \dots v_d, w].$$

We say that  $w * K$  has the **standard orientation induced by  $K$**  if every simplex in  $K$  has the same orientation in both  $K$  and  $w * K$ , and if every simplex in  $(w * K) - K$  has the standard orientation induced by its presimplex.  $\triangle$

**Lemma 5.2.1.** *Let  $K$  be a finite oriented simplicial complex, let  $w$  be a vertex such that  $w * K$  is well-defined, let  $w * K$  have the standard orientation induced by  $K$ , and let  $d \in \mathbb{Z}$  be nonnegative.*

- (1) *For all distinct  $d$ -simplices  $\sigma_1, \sigma_2 \in (w * K) - K$ , we have that  $\sigma_1$  and  $\sigma_2$  are upper adjacent in  $w * K$  iff  $p(\sigma_1)$  and  $p(\sigma_2)$  are upper adjacent in  $K$ . If  $\sigma_1, \sigma_2$  and  $p(\sigma_1), p(\sigma_2)$  are upper adjacent, then  $\sigma_1$  and  $\sigma_2$  are similarly oriented iff  $p(\sigma_1)$  and  $p(\sigma_2)$  are similarly oriented.*
- (2) *For all  $d$ -simplices  $\sigma \in K$  and  $\sigma' \in (w * K) - K$ , we have that  $\sigma$  and  $\sigma'$  are upper adjacent in  $w * K$  iff the coefficient of  $p(\sigma')$  in  $\tilde{\partial}_d(\sigma)$  is nonzero. If  $\sigma$  and  $\sigma'$  are upper adjacent in  $w * K$ , then they are similarly or dissimilarly oriented in  $w * K$  iff the coefficient of  $p(\sigma')$  in  $\tilde{\partial}_d(\sigma)$  is  $-1$ , or  $+1$ , respectively.*

**Proof.** (1) Let  $\sigma_1, \sigma_2 \in (w * K) - K$  be distinct  $d$ -simplices. Suppose  $p(\sigma_1)$  and  $p(\sigma_2)$  are upper adjacent. Then there is a  $d$ -simplex  $\zeta \in K$  containing both  $p(\sigma_1)$  and  $p(\sigma_2)$  as faces. Then  $w * \zeta$  is a  $(d + 1)$ -simplex in  $w * K$ , and it must contain  $w * p(\sigma_1) = \sigma_1$  and  $w * p(\sigma_2) = \sigma_2$  as faces, so  $\sigma_1$  and  $\sigma_2$  are upper adjacent. Now suppose  $\sigma_1$  and  $\sigma_2$  are upper adjacent. Then there is a  $(d + 1)$ -simplex  $\tau \in w * K$  that contains  $\sigma_1$  and  $\sigma_2$  as faces. Note also that  $\tau$  contains  $w$ , so  $\tau \in (w * K) - K$ . Then  $p(\tau)$  is a  $d$ -simplex in  $K$ , and we see that  $p(\tau)$  contains both  $p(\sigma_1)$  and  $p(\sigma_2)$  as faces, so  $p(\sigma_1)$  and  $p(\sigma_2)$  are upper adjacent.

Therefore  $p(\sigma_1)$  and  $p(\sigma_2)$  are upper adjacent iff  $\sigma_1$  and  $\sigma_2$  are upper adjacent. Now we must show that the similarities and dissimilarities match.

Suppose each of  $p(\sigma_1), p(\sigma_2)$  and  $\sigma_1, \sigma_2$  are upper adjacent. By Lemma 3.4.4, we know that each pair  $p(\sigma_1), p(\sigma_2)$  and  $\sigma_1, \sigma_2$  are similarly oriented with respect to their common upper simplex iff the pair share a dissimilar reduced common lower simplex. We will show that  $p(\sigma_1)$  and  $p(\sigma_2)$  share a similar reduced common lower simplex iff  $\sigma_1$  and  $\sigma_2$  share a similar reduced common lower simplex, which will therefore suffice to complete our proof of part (1).

Let

$$p(\sigma_1) = s_1[v_1 v_2 \dots v_{d-1} x] \text{ and } p(\sigma_2) = s_1[v_1 v_2 \dots v_{d-1} y],$$

where  $s_1$  and  $s_2$  are each  $\pm 1$ , depending on orientations. By the definition of standard induced orientation we have that

$$\sigma_1 = (-1)^d s_1[v_1 v_2 \dots v_{d-1} x w] \text{ and } \sigma_2 = (-1)^d s_1[v_1 v_2 \dots v_{d-1} y w].$$

The common reduced lower simplex of  $p(\sigma_1)$  and  $p(\sigma_1)$  is  $\eta = \pm[v_1 v_2 \dots v_{d-1}]$ , with the sign depending on the orientation of  $\eta$ . Since the coefficients of  $\eta$  in  $\tilde{\partial}_{d-1}(p(\sigma_1))$

and  $\tilde{\partial}_{d-1}(p(\sigma_2))$  are  $\pm(-1)^{d-1}s_1$  and  $\pm(-1)^{d-1}s_2$ , respectively, we see that  $\eta$  is a similar reduced common lower simplex of  $p(\sigma_1)$  and  $p(\sigma_2)$  iff  $s_1 = s_2$ . The reduced common lower simplex of  $\sigma_1$  and  $\sigma_2$  is  $\eta' = \pm[v_1v_2 \dots v_{d-1}w]$ , with the sign depending on the orientation of  $\eta'$ . Since the coefficients of  $\eta'$  in  $\tilde{\partial}_d(\sigma_1)$  and  $\tilde{\partial}_d(\sigma_2)$  are  $\pm(-1)^{d-1}(-1)^d s_1 = \mp s_1$  and  $\pm(-1)^{d-1}(-1)^d s_2 = \mp s_2$ , we see that  $\eta'$  is a similar reduced common lower simplex of  $\sigma_1$  and  $\sigma_2$  iff  $s_1 = s_2$ . Therefore  $p(\sigma_1)$  and  $p(\sigma_2)$  share a similar reduced common lower simplex iff  $\sigma_1$  and  $\sigma_2$  share a similar reduced common lower simplex.

(2) Let  $\sigma \in K$  and  $\sigma' \in (w * K) - K$  be  $d$ -simplices. Suppose the coefficient of  $p(\sigma')$  in  $\tilde{\partial}_d(\sigma)$  is nonzero, meaning that  $p(\sigma')$  is a face of  $\sigma$ . Then  $w * \sigma$  is a  $(d + 1)$ -simplex in  $w * K$  that contains both  $\sigma$  and  $w * p(\sigma') = \sigma'$  as faces, so  $\sigma$  and  $\sigma'$  are upper adjacent in  $w * K$ . Suppose  $\sigma$  and  $\sigma'$  are upper adjacent in  $w * K$ . Then  $\sigma$  and  $\sigma'$  must be lower adjacent in  $w * K$ , and there must be some  $(d - 1)$ -simplex that is a face of both  $\sigma$  and  $\sigma'$ . This  $(d - 1)$ -simplex must be in  $K$  and must also be a face of  $\sigma'$ , so it must be  $p(\sigma')$ . Since  $p(\sigma')$  is a proper face of  $\sigma$ , this presimplex must be contained in  $\tilde{\partial}_d(\sigma)$ .

Now we must show that the similarities match up. Suppose  $\sigma$  and  $\sigma'$  are upper adjacent. Then they are reduced lower adjacent with reduced common lower simplex  $p(\sigma')$ . Let  $p(\sigma') = [v_1v_2 \dots v_d]$ . Then  $\sigma' = (-1)^d[v_1v_2 \dots v_dw]$ . Then the coefficient of  $p(\sigma')$  in  $\tilde{\partial}_d(\sigma')$  is  $(-1)^d(-1)^d = +1$ . Since  $p(\sigma')$  is the common lower simplex of  $\sigma$  and  $\sigma'$ , using Lemma 3.4.4 this implies that  $\sigma$  and  $\sigma'$  are upper adjacent in  $w * K$  and similarly, or dissimilarly oriented, iff they are reduced lower adjacent in  $w * K$  with dissimilar, or similar, reduced common lower simplex iff the coefficient of  $p(\sigma')$  in  $\tilde{\partial}_d(\sigma)$  is  $-1$ , or  $+1$ , respectively.  $\square$

Now we are prepared to prove a very important result about cones of simplicial complexes. This theorem was first proved in [DURE, Corollary 4.11], where it is actually a corollary of a much stronger and more general result, but the proof given here is entirely our own, and surprisingly is seen to rest largely on the Reduced Laplacian Matrix Theorem, Theorem 3.4.8. We state this result in our own notation, which looks rather different from the original statement of this result. Recall the definitions of multiset sum and scaled multiset from the definitions at the beginning of Section 2.2.

**Theorem 5.2.2.** *Let  $K$  be a finite simplicial complex, let  $w$  be a vertex such that  $w * K$  is well-defined, and let  $d \in \mathbb{Z}$  be nonnegative. Then*

$$\text{Spec}_{\mathbb{N}\mathbb{Z}}(\tilde{\Delta}_d^{UP}(w * K)) = \{[1]^{f_d(K)}\} +_M \text{Spec}(\tilde{\Delta}_d(K)).$$

**Proof.** Let  $f_d(K) = m$  and  $f_{d-1}(K) = n$ . Let  $\sigma_1, \dots, \sigma_m \in K$  and  $\sigma_{m+1}, \dots, \sigma_{m+n} \in (w * K) - K$  be all the  $d$ -simplices of  $w * K$ , and for all  $i \in \{1, \dots, n\}$  let  $\eta_i = p(\sigma_{m+i})$ . Note that since each  $(d - 1)$ -simplex in  $K$  corresponds to a  $d$ -simplex in  $(w * K) - K$ , the set  $\{\eta_1, \dots, \eta_n\}$  represents all the  $(d - 1)$ -simplices of  $K$ . This proof proceeds in several steps.

STEP (1): First we will demonstrate that

$$\tilde{\mathcal{L}}_d^{UP}(w * K) = \begin{pmatrix} \tilde{\mathcal{L}}_d^{UP}(K) + I_m & \tilde{\mathcal{B}}_d^T(K) \\ \tilde{\mathcal{B}}_d(K) & \tilde{\mathcal{L}}_{d-1}^{UP}(K) \end{pmatrix},$$

where  $I_m$  is the  $m \times m$  identity matrix.

Let  $i, j \in \{1, \dots, m\}$ . Suppose  $i = j$ . Then by the Reduced Laplacian Matrix Theorem, Theorem 3.4.8, the  $i$ th diagonal entry of  $\tilde{\mathcal{L}}_d^{UP}(w * K)$  is the upper degree of  $\sigma_i$  in  $w * K$ . Note that the upper degree of  $\sigma_i$  in  $w * K$  is precisely one more than its upper degree in  $K$ , because there is only one  $(d + 1)$ -simplex in  $(w * K) - K$  that contains  $\sigma_i$ , namely  $w * \sigma_i$ . Suppose  $i \neq j$ . Then by the Reduced Laplacian Matrix Theorem, the  $ij$ th entry of  $\tilde{\mathcal{L}}_d^{UP}(w * K)$  is 0 if  $\sigma_i$  and  $\sigma_j$  are not upper adjacent in  $w * K$ , and  $+1$  or  $-1$  if they are upper adjacent and similarly or dissimilarly oriented, respectively. Note that there cannot be a  $(d + 1)$ -simplex in  $w * K$  containing both  $\sigma_i$  and  $\sigma_j$  that is not in  $K$ , so these two  $d$ -simplices are upper adjacent in  $w * K$  iff they are upper adjacent in  $K$ . Also, by the definition of the induced standard orientation of  $w * K$ , the orientations of  $\sigma_i$  and  $\sigma_j$  in  $K$  and  $w * K$  are identical. Therefore, we see that the  $ij$ th component of  $\tilde{\mathcal{L}}_d^{UP}(w * K)$  is the same as the  $ij$ th component of  $\tilde{\mathcal{L}}_d^{UP}(K)$ . It follows then that the  $m \times m$  upper left block of  $\tilde{\mathcal{L}}_d^{UP}(w * K)$  is identical to  $\tilde{\mathcal{L}}_d^{UP}(K) + I_m$ , where the latter matrix is calculated with respect to the ordered basis  $\{\sigma_1, \dots, \sigma_m\}$ .

Let  $i, j \in \{m + 1, \dots, m + n\}$ . Suppose  $i = j$ . Then by the Reduced Laplacian Matrix Theorem the  $i$ th diagonal entry of  $\tilde{\mathcal{L}}_d^{UP}(w * K)$  is the upper degree of  $\sigma_i$  in  $w * K$ . It is not difficult to see that the number of  $(d + 1)$ -simplices in  $w * K$  containing  $\sigma_i \in (w * K) - K$  as a face is the same as the number of  $d$ -simplices in  $K$  containing  $p(\sigma_i) = \eta_{i-m}$ . Therefore the upper degree of  $\sigma_i$  in  $w * K$  is equal to the upper degree of  $\eta_{i-m}$  in  $K$ . Now, suppose  $i \neq j$ . By Lemma 5.2.1(1), we know that  $\sigma_i$  and  $\sigma_j$  are upper adjacent iff  $p(\sigma_i) = \eta_{i-m}$  and  $p(\sigma_j) = \eta_{j-m}$  are upper adjacent in  $K$ , and that if each pair of simplices is upper adjacent then the pairs are similarly, or dissimilarly, oriented simultaneously. Therefore, we see that the  $n \times n$  lower right block of  $\tilde{\mathcal{L}}_d^{UP}(w * K)$  is identical to  $\tilde{\mathcal{L}}_{d-1}^{UP}(K)$ , where the latter matrix is calculated with respect to the ordered basis  $\{\eta_1, \dots, \eta_n\}$ .

Let  $i \in \{m + 1, \dots, m + n\}$  and  $j \in \{1, \dots, m\}$ . By the Reduced Laplacian Matrix Theorem, Theorem 3.4.8(2), and the definition of the matrix representation of  $\tilde{\partial}_d(K)$ , the  $ij$ th entry of  $\tilde{\mathcal{L}}_d^{UP}(w * K)$  is 0 if  $\sigma_i$  and  $\sigma_j$  are not upper adjacent in  $w * K$ ; and  $+1$  or  $-1$  if they are upper adjacent and similarly or dissimilarly oriented, respectively, which is the case iff the coefficient of  $p(\sigma_i) = \eta_{i-m}$  in  $\tilde{\partial}_d(\sigma_j)$  is  $-1$ , or  $+1$ , respectively, which is the case iff the  $ij$ th entry of  $\tilde{\mathcal{B}}_d(K)$  is  $-1$  or  $+1$ , respectively, where  $\tilde{\mathcal{B}}_d(K)$  is the matrix of  $\tilde{\partial}_d(K)$  with respect to the ordered bases  $\{\sigma_1, \dots, \sigma_m\}$  and  $\{\eta_1, \dots, \eta_n\}$  for the  $d$ -chains and  $(d - 1)$ -chains of  $K$ , respectively. Therefore, it follows that the lower left block of  $\tilde{\mathcal{L}}_d^{UP}(w * K)$  is identical to  $\tilde{\mathcal{B}}_d(K)$ . Since  $\tilde{\mathcal{L}}_d^{UP}(w * K)$  is a symmetric matrix, this implies that the upper right block of  $\tilde{\mathcal{L}}_d^{UP}(w * K)$  is identical to  $\tilde{\mathcal{B}}_d^T(K)$ .

STEP (2): (In this part, we will be doing many calculations with block matrices and block vectors. The reader should verify that the dimensions of appropriate blocks match up throughout our calculations.) Let

$$X = \begin{pmatrix} \tilde{\mathcal{L}}_d^{UP} + I_m & -\tilde{\mathcal{B}}_d^T \\ -\tilde{\mathcal{B}}_d & \tilde{\mathcal{L}}_{d-1}^{UP} \end{pmatrix} = \begin{pmatrix} \tilde{\mathcal{B}}_{d+1} \tilde{\mathcal{B}}_{d+1}^T + I_m & -\tilde{\mathcal{B}}_d^T \\ -\tilde{\mathcal{B}}_d & \tilde{\mathcal{B}}_d \tilde{\mathcal{B}}_d^T \end{pmatrix}$$

and

$$Y = \begin{pmatrix} \tilde{\mathcal{L}}_d + I_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \tilde{\mathcal{B}}_{d+1} \tilde{\mathcal{B}}_{d+1}^T + \tilde{\mathcal{B}}_d^T \tilde{\mathcal{B}}_d + I_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

(For simplicity, in this step we will omit the arguments from our matrix symbols, and assume that every matrix is with respect to  $K$ .) We will demonstrate that  $\text{Spec}(X) = \text{Spec}(Y)$ .

Let

$$P = \begin{pmatrix} I_m & \tilde{\mathcal{B}}_d^T \\ -\tilde{\mathcal{B}}_d & I_n \end{pmatrix}.$$

Keeping in mind that  $\tilde{\partial}_i \tilde{\partial}_{i+1}$  is the zero transformation, and hence  $\tilde{\mathcal{B}}_i \tilde{\mathcal{B}}_{i+1}$  is the zero matrix, for all nonnegative integers  $i$ , the reader should verify that

$$PX = YP = \begin{pmatrix} \tilde{\mathcal{B}}_{d+1} \tilde{\mathcal{B}}_{d+1}^T + \tilde{\mathcal{B}}_d^T \tilde{\mathcal{B}}_d + I_m & \tilde{\mathcal{B}}_d^T \tilde{\mathcal{B}}_d \tilde{\mathcal{B}}_d^T + \tilde{\mathcal{B}}_d^T \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Now if we can show that  $P$  is invertible, then it follows that  $X = P^{-1}YP$ . From [FIS97, Exercise 12(a), page 249], it would then follow that  $\text{Spec}(X) = \text{Spec}(Y)$ . We prove that  $P$  is invertible indirectly. Let

$$Q = \begin{pmatrix} I_m & -\tilde{\mathcal{B}}_d^T \\ \tilde{\mathcal{B}}_d & I_n \end{pmatrix}.$$

Then

$$\begin{aligned} PQ &= \begin{pmatrix} I_m & \tilde{\mathcal{B}}_d^T \\ -\tilde{\mathcal{B}}_d & I_n \end{pmatrix} \begin{pmatrix} I_m & -\tilde{\mathcal{B}}_d^T \\ \tilde{\mathcal{B}}_d & I_n \end{pmatrix} = \begin{pmatrix} \tilde{\mathcal{B}}_d^T \tilde{\mathcal{B}}_d + I_m & \mathbf{0} \\ \mathbf{0} & \tilde{\mathcal{B}}_d \tilde{\mathcal{B}}_d^T + I_n \end{pmatrix} \\ &= \begin{pmatrix} \tilde{\mathcal{B}}_d^T \tilde{\mathcal{B}}_d & \mathbf{0} \\ \mathbf{0} & \tilde{\mathcal{B}}_d \tilde{\mathcal{B}}_d^T \end{pmatrix} + I_{m+n} = \begin{pmatrix} \tilde{\mathcal{B}}_d^T & \mathbf{0} \\ \mathbf{0} & \tilde{\mathcal{B}}_d \end{pmatrix} \begin{pmatrix} \tilde{\mathcal{B}}_d & \mathbf{0} \\ \mathbf{0} & \tilde{\mathcal{B}}_d^T \end{pmatrix} + I_{m+n} \\ &= \begin{pmatrix} \tilde{\mathcal{B}}_d^T & \mathbf{0} \\ \mathbf{0} & \tilde{\mathcal{B}}_d \end{pmatrix} \begin{pmatrix} \tilde{\mathcal{B}}_d^T & \mathbf{0} \\ \mathbf{0} & \tilde{\mathcal{B}}_d \end{pmatrix}^T + I_{m+n}. \end{aligned}$$

Label the first term in the last equality  $A$ . By Lemma 2.2.8 we know that  $A$  represents a positive semidefinite linear operator, so by Lemma 2.2.7 the eigenvalues of  $A$  are nonnegative. Note also that adding  $I_{m+n}$  to  $A$  has the effect of adding 1 to each eigenvalue, so that  $\text{Spec}(A + I_{m+n}) = \{1, 1, \dots, 1\} +_M \text{Spec}(A)$ . Therefore the eigenvalues of  $PQ = A + I_{m+n}$  are strictly positive. This means that 0 is not an eigenvalue of  $PQ$ , which means that the nullspace of the operator represented by  $PQ$  is trivial, which implies that  $PQ$  is invertible. Then

$$\det(P) \det(Q) = \det(PQ) \neq 0,$$

so  $\det(P) \neq 0$ , so  $P$  is invertible.

STEP (3): Putting Steps (1) and (2) together, we have proven that

$$\text{Spec}(\Delta_d^{UP}(w * K)) = \text{Spec}(X) = \text{Spec}(Y) = \{[0]^n\} \cup_M \left( \{[1]^m\} +_M \text{Spec}(\tilde{\Delta}_d(K)) \right).$$

Since the elements of  $\{[1]^m\} +_M \text{Spec}(\tilde{\Delta}_d(K))$  are strictly positive, this implies that

$$\text{Spec}_{NZ}(\tilde{\Delta}_d^{UP}(w * K)) = \{[1]^{f_d(K)}\} +_M \text{Spec}(\tilde{\Delta}_d(K)).$$

□

From Lemma 4.1.7 and Theorem 5.2.2 we immediately have the following corollary.

**Corollary 5.2.3.** *Let  $K$  be a finite simplicial complex, let  $w$  be a vertex such that  $w * K$  is well-defined, and let  $d \in \mathbb{Z}$  be nonnegative. Then*

$$\text{Spec}_{NZ}(\tilde{\Delta}_d(w * K)) = \{[1]^{f_d(K)+f_{d-1}(K)}\}_{+M} \left( \text{Spec}(\tilde{\Delta}_d(K)) \cup_M \text{Spec}(\tilde{\Delta}_{d-1}(K)) \right).$$

In the case of 2-dimensional cones, Theorem 5.2.2 leads to some truly fascinating results.

**Theorem 5.2.4.** *Let  $K$  be a finite simplicial cone of dimension less than or equal to 2, with base  $G \subseteq K$ . Then*

$$\text{Spec}_{NZ}(\Delta_0(K)) = \left[ \{[1]^{f_0(G)-1}\}_{+M} \left( \text{Spec}_{NZ}(\Delta_0(G)) \right)_{f_0(G)-1} \right] \cup_M \{f_0(G) + 1\} \quad (5.2.1)$$

and

$$\text{Spec}(\Delta_2(K)) = \{[1]^{f_1(G)}\}_{+M} \left( \text{Spec}_{NZ}(\Delta_0(G)) \right)_{f_1(G)}. \quad (5.2.2)$$

**Proof.** Let  $v$  be a vertex of  $K$  such that  $K = v * G$ . Note that  $f_1(G) = f_2(K)$  and  $f_0(G) + 1 = f_0(K)$ .

We know that  $\Delta_0(K) = \Delta_0^{UP}(K) = \tilde{\Delta}_0^{UP}(K)$ , so by Theorem 5.2.2 and Lemma 4.1.7 we have

$$\begin{aligned} \text{Spec}_{NZ}(\Delta_0(K)) &= \{[1]^{f_0(G)}\}_{+M} \text{Spec}(\tilde{\Delta}_0(G)) \\ &= \{[1]^{f_0(G)}\}_{+M} \left( \text{Spec}_{NZ}(\tilde{\Delta}_0^{DN}(G)) \cup_M \text{Spec}_{NZ}(\tilde{\Delta}_0^{UP}(G)) \right)_{f_0(G)}. \end{aligned}$$

It follows from Lemma 2.2.9 that  $\text{Spec}_{NZ}(\tilde{\Delta}_0^{DN}(G)) = \text{Spec}_{NZ}(\mathbb{U}_{f_0(G)}) = \{f_0(G)\}$ , and as before we see that  $\text{Spec}_{NZ}(\tilde{\Delta}_0^{UP}(G)) = \text{Spec}_{NZ}(\Delta_0(G))$ . Therefore

$$\begin{aligned} \text{Spec}_{NZ}(\Delta_0(K)) &= \{[1]^{f_0(G)}\}_{+M} \left( \text{Spec}_{NZ}(\Delta_0(G)) \cup_M \{f_0(G)\} \right)_{f_0(G)} \\ &= \left[ \{[1]^{f_0(G)-1}\}_{+M} \left( \text{Spec}_{NZ}(\Delta_0(G)) \right)_{f_0(G)-1} \right] \cup_M \{f_0(G) + 1\}. \end{aligned}$$

Since  $K$  is at most 2-dimensional, we know  $\Delta_2(K) = \Delta_2^{DN}(K)$ , and also that  $\text{Spec}_{NZ}(\Delta_2^{DN}(K)) = \text{Spec}_{NZ}(\Delta_1^{UP}(K)) = \text{Spec}_{NZ}(\tilde{\Delta}_1^{UP}(K))$ . Therefore by Theorem 5.2.2 it follows that

$$\begin{aligned} \text{Spec}_{NZ}(\Delta_2(K)) &= \{[1]^{f_1(G)}\}_{+M} \text{Spec}(\tilde{\Delta}_1(G)) \\ &= \{[1]^{f_1(G)}\}_{+M} \left( \text{Spec}_{NZ}(\tilde{\Delta}_1^{DN}(G)) \cup_M \text{Spec}_{NZ}(\tilde{\Delta}_1^{UP}(G)) \right)_{f_1(G)}. \end{aligned}$$

Lemma 4.1.8 and some simplification yield that  $\text{Spec}_{NZ}(\tilde{\Delta}_1^{DN}(G)) = \text{Spec}_{NZ}(\tilde{\Delta}_0^{UP}(G)) = \text{Spec}_{NZ}(\Delta_0(G))$  and  $\text{Spec}_{NZ}(\tilde{\Delta}_1^{UP}(G)) = \text{Spec}_{NZ}(\tilde{\Delta}_2^{DN}(G)) = \text{Spec}_{NZ}(\Delta_2^{DN}(G)) = \text{Spec}_{NZ}(\Delta_2(G))$ . Furthermore, since  $K$  is at most 2-dimensional it must be the case that  $G$  is at most 1-dimensional, so  $\Delta_2(G)$  is a zero matrix and  $\text{Spec}_{NZ}(\Delta_2(G)) = \emptyset$ . Finally, Theorem 4.1.4 implies that  $H_2(K; \mathbb{R}) \cong N(\Delta_2(K))$ , and since  $H_2(K; \mathbb{R})$  is zero dimensional

since  $K$  is a cone, we know  $0$  is not in  $\text{Spec}(\Delta_2(K))$ , so  $\text{Spec}_{NZ}(\Delta_2(K)) = \text{Spec}(\Delta_2(K))$ . Therefore

$$\text{Spec}(\Delta_2(K)) = \{[1]^{f_1(G)}\}_{+M} \left( \text{Spec}_{NZ}(\Delta_0(G)) \right)_{f_1(G)}.$$

□

**Theorem 5.2.5.** *Let  $K$  be a finite simplicial cone of dimension less than or equal to 2.*

(i) *If  $f_0(K) \geq f_2(K) + 2$ , then*

$$\text{Spec}(\Delta_0(K)) = \text{Spec}(\Delta_2(K)) \cup_M \{0, [1]^{f_0(K)-f_2(K)-2}, f_0(K)\}.$$

(ii) *If  $f_0(K) < f_2(K) + 2$ , then*

$$\text{Spec}(\Delta_0(K)) \cup_M \{[1]^{f_2(K)-f_0(K)+2}\} = \text{Spec}(\Delta_2(K)) \cup_M \{0, f_0(K)\}.$$

**Proof.** Let  $K = v * G$ . Note that  $f_0(G) + 1 = f_0(K)$  and  $f_1(G) = f_2(K)$ .

For part (i), suppose  $f_0(K) \geq f_2(K) + 2$ . Then  $f_0(K) - 2 \geq f_2(K)$ . Then from Equations 5.2.1 and 5.2.2 of Theorem 5.2.4, and recalling that  $f_0(G) + 1 = f_0(K)$  and  $f_1(G) = f_2(K)$ , we have

$$\begin{aligned} \text{Spec}_{NZ}(\Delta_0(K)) &= \left[ \{[1]^{f_0(K)-2}\}_{+M} \left( \text{Spec}_{NZ}(\Delta_0(G)) \right)_{f_0(K)-2} \right] \cup_M \{f_0(K)\} \\ &= \left[ \{[1]^{f_2(K)}\}_{+M} \left( \text{Spec}_{NZ}(\Delta_0(G)) \right)_{f_2(K)} \right] \\ &\quad \cup_M \left[ \{[1]^{f_0(K)-f_2(K)-2}\}_{+M} \{[0]^{f_0(K)-f_2(K)-2}\} \right] \cup_M \{f_0(K)\} \\ &= \text{Spec}(\Delta_2(K)) \cup_M \{[1]^{f_0(K)-f_2(K)-2}, f_0(K)\}. \end{aligned}$$

The last expression in this chain of equalities has exactly  $f_2(K) + f_0(K) - f_2(K) - 2 + 1 = f_0(K) - 1$  elements. Hence we conclude that

$$\text{Spec}(\Delta_0(K)) = \text{Spec}(\Delta_2(K)) \cup_M \{0, [1]^{f_0(K)-f_2(K)-2}, f_0(K)\}.$$

For part (ii), suppose  $f_0(K) < f_2(K) + 2$ . Then  $f_2(K) > f_0(K) - 2$ . Also, from Theorem 4.1.4 we know  $0 \in \text{Spec}(\Delta_0(G))$ , so  $\text{Spec}_{NZ}(\Delta_0(G))$  has no more than  $f_0(G) - 1 = f_0(K) - 2 < f_2(K)$  elements, so at least  $f_2(K) - (f_0(K) - 2) = f_2(K) - f_0(K) + 2$  elements of  $\left( \text{Spec}_{NZ}(\Delta_0(G)) \right)_{f_2(K)}$  are 0. Then from Equations 5.2.1 and 5.2.2 of Theorem 5.2.4 we have

$$\begin{aligned} \text{Spec}(\Delta_2(K)) &= \{[1]^{f_2(K)}\}_{+M} \left( \text{Spec}_{NZ}(\Delta_0(G)) \right)_{f_2(K)} \\ &= \left[ \{[1]^{f_0(K)-2}\}_{+M} \left( \text{Spec}_{NZ}(\Delta_0(G)) \right)_{f_0(K)-2} \right] \\ &\quad \cup_M \left[ \{[1]^{f_2(K)-f_0(K)+2}\}_{+M} \{[0]^{f_2(K)-f_0(K)+2}\} \right] \\ &= [\text{Spec}_{NZ}(\Delta_0(K)) - \{f_0(K)\}] \cup_M \{[1]^{f_2(K)-f_0(K)+2}\}. \end{aligned}$$



The last expression in this chain of equalities has exactly  $|Spec_{NZ}(\Delta_0(K))| - 1 + f_2(K) - f_0(K) + 2 = |Spec_{NZ}(\Delta_0(K))| + f_2(K) - f_0(K) + 1$  elements, and since this must be the same number as  $f_2(K)$ , we see that  $|Spec_{NZ}(\Delta_0(K))| = f_0(K) - 1$ , so  $Spec_{NZ}(\Delta_0(K)) = Spec(\Delta_0(K)) - \{0\}$ . Therefore

$$Spec(\Delta_0(K)) \cup_M \{[1]^{f_2(K)-f_0(K)+2}\} = Spec(\Delta_2(K)) \cup_M \{0, f_0(K)\}.$$

□

We will now discuss some families of cones. If we take the cone of a polygonal circle, the result is called a **pie**. This object is so-named because it looks like a pie in the shape of a polygon cut into slices. The cone on a polygonal arc is called a **fan**. The cone on a set of disjoint edges is called a **pinwheel**. The cone on a set of disjoint points is called an **asterisk**. Finally, the cone of an asterisk is a flapwheel, so in a sense a flapwheel is a sort of double cone.

In Figure 5.2.2, we have (i) a 6-pie, (ii) 4-fan, (iii) a 3-pinwheel, (iv) a 5-asterisk, and (v) a 5-flapwheel. In each picture, the labeled vertex is the coning vertex.

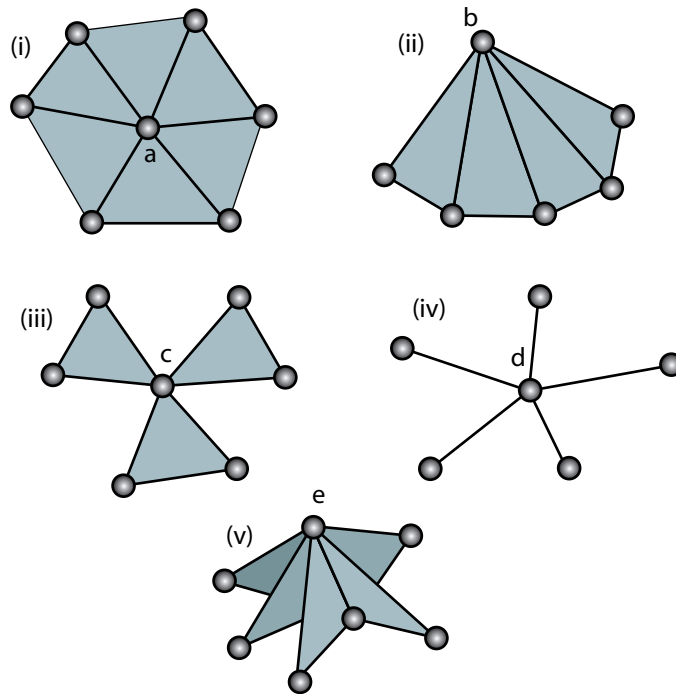


Figure 5.2.2.

Since all of these objects are cones, Theorem 5.2.5 tells us that essentially all of their Laplacian spectra information is contained in the graph theory spectrum of these objects. Beyond this, the spectra of pies and fans seem to show few decipherable patterns. However, it turns out that the Laplacian spectra of pinwheels, asterisks, and flapwheels are completely determined by Theorems 5.2.5 and 5.1.8.

**Lemma 5.2.6.** *Let  $P$  be a pinwheel, and let  $f_2(P) = n > 0$ . Then  $\text{Spec}(\Delta_0(P)) = \{0, [1]^{n-1}, [3]^n, 2n + 1\}$  and  $\text{Spec}(\Delta_1(P)) = \{[1]^{n-1}, [3]^{2n}, 2n + 1\}$  and  $\text{Spec}(\Delta_2(P)) = \{[3]^n\}$ .*

**Proof.** Note first that  $f_0(P) = 2f_2(P) + 1 = 2n + 1$  and  $f_1(P) = 3n$ . It is easy to see that for all positive integers  $m$  the inequality  $2m \geq m + 1$  holds. That  $P$  is nontrivial means that  $n \geq 1$ , so  $f_0(P) = 2n + 1 \geq n + 2 = f_2(P) + 2$ . Hence by Theorem 5.2.5 we have

$$\text{Spec}(\Delta_0(P)) = \text{Spec}(\Delta_2(P)) \cup_M \{0, [1]^{n-1}, 2n + 1\}.$$

None of the 2-simplices of  $P$  are upper adjacent to each other, so by Theorem 3.3.4 we see that  $\mathcal{L}_2(P) = 3I_n$ , where  $I_n$  is the  $n \times n$  identity matrix, so  $\text{Spec}(\Delta_2(P)) = \{[3]^n\}$ . Therefore  $\text{Spec}(\Delta_0(P)) = \{0, [1]^{n-1}, [3]^n, 2n + 1\}$ .

The statement about  $\text{Spec}(\Delta_1(P))$  follows from Corollary 4.1.9, and the fact that  $f_1(P) = 3n$ .  $\square$

**Lemma 5.2.7.** *Let  $A$  be an asterisk, and let  $f_0(A) = n > 0$ . Then  $\text{Spec}(\Delta_0(A)) = \{0, [1]^{n-2}, n\}$  and  $\text{Spec}(\Delta_1(A)) = \{[1]^{n-2}, n\}$ .*

**Proof.** Since  $A$  contains no 2-simplices, by Theorem 5.2.5 we have  $\text{Spec}(\Delta_0(A)) = \text{Spec}(\Delta_2(P)) \cup_M \{0, [1]^{n-2}, n\} = \{0, [1]^{n-2}, n\}$ . Since  $A$  is 1-dimensional, we have  $\Delta_1(A) = \Delta_1^{DN}(A)$ . Since  $\Delta_0(A) = \Delta_0^{UP}(A)$  and  $A$  has one fewer edges than vertices, we see by Lemma 4.1.8 that  $\text{Spec}(\Delta_1(A)) = \text{Spec}_{NZ}(\Delta_0(A)) = \{[1]^{n-2}, n\}$ .  $\square$

Viewing flapwheels as simplicial cones, we now have another proof of Theorem 5.1.6.

**Alternate proof of Theorem 5.1.6.** As a reminder, we have a flapwheel  $F$  with  $n > 1$  flaps. As noted above, we see that  $F$  is combinatorically equivalent to a cone on an asterisk  $A$  with  $n + 1$  vertices. By Equation 5.2.1 from the proof of Theorem 5.2.4, and Lemma 5.2.7 above, we have

$$\begin{aligned} \text{Spec}_{NZ}(\Delta_0(F)) &= \left[ \{[1]^{f_0(F)-2}\}_{+M} \left( \text{Spec}_{NZ}(\Delta_0(A)) \right)_{f_0(F)-2} \right] \cup_M \{f_0(F)\} \\ &= \left[ \{[1]^n\}_{+M} \left( \text{Spec}_{NZ}(\Delta_0(A)) \right)_n \right] \cup_M \{n + 2\} \\ &= \left[ \{[1]^n\}_{+M} \left( \{[1]^{n-1}, n + 1\} \right)_n \right] \cup_M \{n + 2\} \\ &= \left[ \{[1]^n\}_{+M} \{[1]^{n-1}, n + 1\} \right] \cup_M \{n + 2\} \\ &= \{[2]^{n-1}, n + 2\} \cup_M \{n + 2\} \\ &= \{[2]^{n-1}, n + 2, n + 2\}. \end{aligned}$$

By counting, we see that we are short by a single eigenvalue to get from  $\text{Spec}_{NZ}(\Delta_0(F))$  to  $\text{Spec}(\Delta_0(F))$ , so  $\text{Spec}(\Delta_0(F)) = \{0, [2]^{n-1}, n + 2, n + 2\}$ .

Now, by Equation 5.2.2 from Theorem 5.2.4, and Proposition 5.2.7 above, we have

$$\begin{aligned} \text{Spec}(\Delta_2(F)) &= \{[1]^{f_2(F)}\}_{+M} \left( \text{Spec}_{NZ}(\Delta_0(A)) \right)_{f_2(F)} \\ &= \{[1]^n\}_{+M} \left( \{[1]^{n-1}, n + 1\} \right)_n \\ &= \{[2]^{n-1}, n + 2\}. \end{aligned}$$

As mentioned at the end of the original proof of Theorem 5.1.6, the result about  $\text{Spec}(\Delta_1(F))$  follows directly from our results about  $\text{Spec}(\Delta_0(F))$  and  $\text{Spec}(\Delta_2(F))$ .  $\square$

Finally, since a simplex is itself a simplicial complex, we are able to use our results about cones to prove the following tidy fact about the Laplacian spectra of simplices.

**Proposition 5.2.8.** *Let  $n \in \mathbb{Z}$  be nonnegative, and let  $\sigma_n$  denote the  $n$ -simplex. Then*

$$\text{Spec}(\tilde{\Delta}_i(\sigma_n)) = \{[n+1]^{f_i(\sigma_n)}\}$$

for all nonnegative  $i \in \mathbb{Z}$ .

**Proof.** The key observation is that for each nonnegative  $n \in \mathbb{Z}$  we have that  $\sigma_{n+1}$  is combinatorially equivalent to a cone of  $\sigma_n$ . We then prove this proposition by induction on  $n$ .

If  $n = 0$ , then we have  $\sigma_0$  is a single vertex, so  $\tilde{\mathcal{L}}_0(\sigma_0) = (1)$ , so  $\text{Spec}(\tilde{\Delta}_0(\sigma_0)) = \{1\}$ . Suppose  $n \in \mathbb{Z}$  is positive such that the proposition holds for  $\sigma_{n-1}$ . Let  $i \in \mathbb{Z}$  be nonnegative. Then  $\sigma_n$  is a cone of  $\sigma_{n-1}$ , so by Theorem 5.2.2 and the induction hypothesis we have

$$\begin{aligned} \text{Spec}_{NZ}(\tilde{\Delta}_i^{UP}(\sigma_n)) &= \{[1]^{f_i(\sigma_{n-1})}\} +_M \left( \text{Spec}_{NZ}(\tilde{\Delta}_i(\sigma_{n-1})) \right)_{f_i(\sigma_{n-1})} \\ &= \{[1]^{f_i(\sigma_{n-1})}\} +_M \{[n]^{f_i(\sigma_{n-1})}\} \\ &= \{[n+1]^{f_i(\sigma_{n-1})}\}. \end{aligned}$$

By a completely similar argument we see that

$$\text{Spec}_{NZ}(\tilde{\Delta}_{i-1}^{UP}(\sigma_n)) = \{[n+1]^{f_{i-1}(\sigma_{n-1})}\}.$$

By the work in the preceding paragraph and Lemmas 4.1.7 and 4.1.8 we have that

$$\begin{aligned} \text{Spec}_{NZ}(\tilde{\Delta}_i(\sigma_n)) &= \text{Spec}_{NZ}(\tilde{\Delta}_i^{UP}(\sigma_n)) \cup_M \text{Spec}_{NZ}(\tilde{\Delta}_i^{DN}(\sigma_n)) \\ &= \text{Spec}_{NZ}(\tilde{\Delta}_i^{UP}(\sigma_n)) \cup_M \text{Spec}_{NZ}(\tilde{\Delta}_{i-1}^{UP}(\sigma_n)) \\ &= \{[n+1]^{f_i(\sigma_{n-1})}\} \cup_M \{[n+1]^{f_{i-1}(\sigma_{n-1})}\} \\ &= \{[n+1]^{f_i(\sigma_{n-1})+f_{i-1}(\sigma_{n-1})}\}. \end{aligned}$$

Observe that by a standard combinatorial identity

$$f_i(\sigma_{n-1}) + f_{i-1}(\sigma_{n-1}) = \binom{n}{i+1} + \binom{n}{i} = \binom{n+1}{i+1} = f_i(\sigma_n).$$

Since  $\text{Spec}(\tilde{\Delta}_i(\sigma_n))$  has exactly  $f_i(\sigma_n)$  elements, it follows that

$$\text{Spec}(\tilde{\Delta}_i(\sigma_n)) = \text{Spec}_{NZ}(\tilde{\Delta}_i(\sigma_n)) = \{[n+1]^{f_i(\sigma_{n-1})+f_{i-1}(\sigma_{n-1})}\} = \{[n+1]^{f_i(\sigma_n)}\}.$$

$\square$

# 6

## Directions for Further Research

Clearly there are many open questions concerning Laplacians of simplicial complexes. For one thing, all of the results presented in this project, with the exception of the Combinatorial Hodge Theorem, Theorem 4.1.4, work in the direction of predicting parts of the Laplacian spectra by knowing about the structure of the underlying simplicial complex. What would probably be more interesting and useful, although certainly much harder, would be to be able to predict the structure of a simplicial complex from its Laplacian spectra. This seems like an extremely difficult problem, and we have no immediate intuition as to a good place to start on it.

A somewhat related question, although probably much easier, is to find two simplicial complexes that are combinatorially distinct, but whose Laplacian spectrum is the same in all dimensions. In all of the examples examined, we failed to find any such example. This seems much more like poor luck than evidence that the Laplacian spectra of a simplicial complex is unique.

The really fascinating next direction to take research in this area, from the author's point of view, would be to develop some form of weighted Laplacian operator. This could be done by forming weighted boundary maps for the vector spaces of chains of the complex, and then defining the weighted Laplacian from these. This has been done and very much studied in the case of graphs, where there is the choice of having either vertex or edge weights. (See [CHU96], for instance.) The hope is that by being careful and clever enough, some metric information could be insinuated into the boundary maps, so that the weighted Laplacian spectra might reflect the geometry of a specific simplicial complex.

Ideally, the study of weighted Laplacian spectra of simplicial complex could lead to a number of important things. Hopefully, many of the results developed in this paper might have neat generalizations for the weighted case, with the specifics of the generalizations having something to do with geometric properties. Also, the usefulness and accuracy of simplicial complexes as models of smooth manifolds or the like in many cases depends on specifying the particular geometry of the simplicial complex. Hence any geometric infor-

mation from the weighted Laplacian spectra could be beneficial in these cases.

This is the end. Thank you for reading this.

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